(Problem 13) Show that if G is a finite, planar graph where every vertex has degree exactly 3, then G contains at least two distinct cycles of length less than six.

(Problem 14) Carefully improve the bound of two in your prove.

I will solve only the second problem, as it solves the first also.

First of all we can consider only connected graphs as each component of a non connected graph is a connected graph that has the same property also.

We will use the Euler formula

$$V - 1 + (F - C) = E$$

and in this case $C = 1$. As each vertex is common to three edges and each edge is common to two vertices, we have that

$$3V = 2E$$

and the Euler formula (multiplying it by two) becomes

$$2E - 6 + 3F = 3E$$

and then

$$E = 3F - 6 \tag{1}$$

Let N_i be the length of a cycle around face i ($1 \leq i \leq F$), and suppose that exactly and only the first k faces the cycle are of length lower than six. Then

$$\sum_{i=1}^{F} N_i \geq 3k + 6(F - k) \tag{2}$$

(the second member indicates that the first k cycles are of length of at least 3, the others of length of at least 6).

Note that the above sum is twice the value of E (each vertex is counted twice as it is common to two faces). So the second member of 1 is greater of the half of the second member of 2, and we get

$$6F - 12 \geq 6F - 3k.$$

Simplifying it we obtain that $k \geq 4$.

So the minimum number of cycles of length less than six is 4. We cannot improve the bound more than this as graph K_4 is a graph with exactly four nodes, all with degree three and each face has a cycle of length 3.