

Logic Programs with Propositional Connectives and Aggregates

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Abstract

Answer set programming (ASP) is a logic programming paradigm that can be used to solve complex combinatorial search problems. Aggregates are an ASP construct that plays an important role in many applications. Defining a satisfactory semantics of aggregates turned out to be a difficult problem, and in this paper we propose a new approach, based on an analogy between aggregates and propositional connectives. First, we extend the definition of an answer set/stable model to cover arbitrary propositional theories; then we define aggregates on top of them both as primitive constructs and as abbreviations for formulas. Our definition of an aggregate combines expressiveness and simplicity, and it inherits many theorems about programs with nested expressions, such as theorems about strong equivalence and splitting.

1 Introduction

Answer set programming (ASP) is a logic programming paradigm that can be used to solve complex combinatorial search problems ([Marek and Truszczyński, 1999]), ([Niemelä, 1999]). ASP is based on the stable model semantics [Gelfond and Lifschitz, 1988] for logic programs: programming in ASP consists in writing a logic program whose stable models (also called answer sets) represent the solution to our problem. ASP has been used, for instance, in planning [Dimopoulos *et al.*, 1997; Lifschitz, 1999], model checking [Liu *et al.*, 1998; Heljanko and Niemelä, 2001], product configuration [Soininen and Niemelä, 1998], logical cryptanalysis [Hietalahti *et al.*, 2000], workflow specification [Trajcevski *et al.*, 2000; Koksäl

et al., 2001], reasoning about policies [Son and Lobo, 2001], wire routing problems [Erdem *et al.*, 2000] and phylogeny reconstruction problems [Erdem *et al.*, 2003].

The stable models of a logic program are found by systems called *answer set solvers*. Answer set solvers can be considered the equivalent of SAT solvers — systems used to find the models of propositional formulas — in logic programming. On the other hand, it is much easier to express, in logic programming, recursive definitions (such as reachability in a graph) and defaults. Several answer set solvers have been developed so far, with SMODELS¹ and DLV² among the most popular. As in the case of SAT solvers, answer set solver competitions — where answer set solvers are compared to each others in terms of performance — are planned to be held regularly.³

An important construct in ASP are aggregates. Aggregates allow, for instance, to perform set operations such as counting the number of atoms in a set that are true, or summing weights the weights of the atoms that are true. We can, for instance, express that a node in a graph has exactly one color by the following cardinality constraint:

$$1 \leq \{c(\text{node}, \text{color}_1), \dots, c(\text{node}, \text{color}_m)\} \leq 1.$$

As another example, a weight constraint of the form

$$3 \leq \{p = 1, q = 2, r = 3\} \tag{1}$$

intuitively says that the sum of the weights (the numbers after the “=” sign) of the atoms from the list p, q, r that are true is at least 3.

Aggregates are a hot topic in ASP not only because of their importance, but also because there is no standard understanding of the concept of an aggregate. In fact, different answer set solvers implement different definitions of aggregates: for instance, SMODELS implements cardinality and weight constraints [Niemelä and Simons, 2000], while DLV implements aggregates as defined by Faber, Leone and Pfeifer (2005) (we call them FLP-aggregates). Unfortunately, constructs that are intuitively equivalent to each other may actually lead to different stable models. In some sense, no current definition of an aggregate can be considered fully satisfactory, as each of them seems to have properties that look unintuitive. For instance, it is somehow puzzling that, as noticed in [Ferraris and Lifschitz, 2005b], weight constraints

$$0 \leq \{p = 2, p = -1\} \quad \text{and} \quad 0 \leq \{p = 1\}$$

¹<http://www.tcs.hut.fi/Software/smodels/>

²<http://www.dbai.tuwien.ac.at/proj/dlv/>

³<http://asparagus.cs.uni-potsdam.de/contest/>

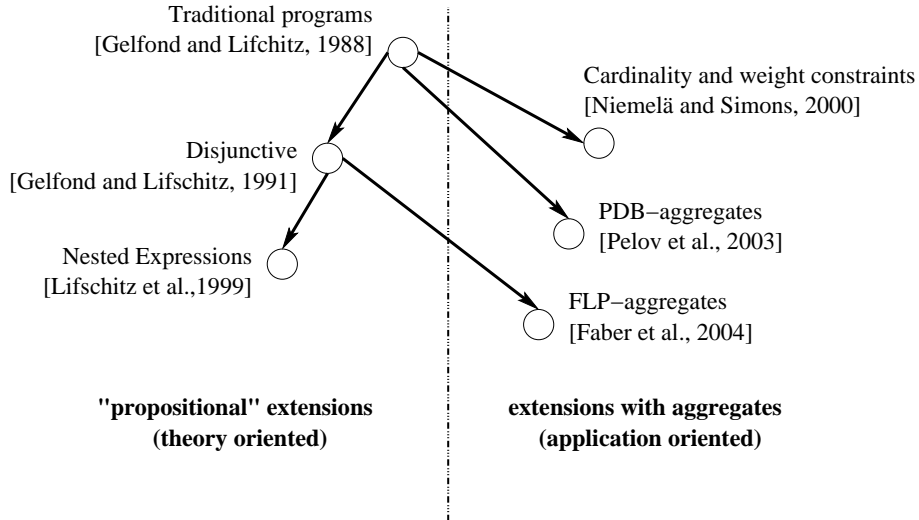


Figure 1: Evolution of the stable model semantics.

are semantically different from each other (may lead to different stable models). Part of this problem is probably related to the lack of mathematical tools for studying properties of programs with aggregates, in particular for reasoning about the correctness of programs with aggregates.

This paper addresses the problems of aggregates mentioned above by (i) giving a new semantics of aggregates that, we argue, is more satisfactory than the existing alternatives, and (ii) providing tools for studying properties of logic programs with aggregates.

Our approach is based on a relationship between two directions of research on extending the stable model semantics: the work on aggregates, mentioned above, and the work on “propositional extensions” (see Figure 1). The latter makes the syntax of rules more and more similar to the syntax of propositional formulas. In disjunctive programs, the head of each rule is a (possibly empty) disjunction of atoms, while in programs with nested expressions the head and body of each rule can be any arbitrary formula built with connectives AND, OR and NOT. For instance,

$$\neg(p \vee \neg q) \leftarrow p \vee \neg \neg r$$

is a rule with nested expressions. Programs with nested expressions are quite attractive especially relative to point (ii) above, because many theorems about properties of logic programs have been proved for programs of this kind. For instance, the splitting set theorem [Lifschitz and Turner, 1994; Erdoğan and Lifschitz, 2004]

simplifies the task of computing the stable models of a program/theory by breaking it into two parts. Work on strong equivalence [Lifschitz *et al.*, 2001] allows us to modify a program/theory with the guarantee that stable models are preserved (more details in Section 2.4).

Nested expressions have already been used to express aggregates: [Ferraris and Lifschitz, 2005b] showed that each weight constraint can be replaced by a nested expressions, preserving its stable models. As a consequence, theorems about nested expressions can be used for programs with weight constraints. It turns out, however, that nested expressions are not sufficiently general for defining a semantics for aggregates that overcomes the unintuitive features of the existing approaches. For this reason, we extend the syntax of rules with nested expressions, allowing implication in every part of a “rule”, and not only as the outermost connective. (We understand a rule as an implication from the body to the head). A “rule” is then an arbitrary propositional formula, and a program an arbitrary propositional theory. Our new definition of a stable model, like all the other definitions, is based on the process of constructing a reduct. The process that we use looks very different from all the others, and in particular for programs with nested expressions. Nevertheless, it turns out that in application to programs with nested expressions, our definition is equivalent to the one from [Lifschitz *et al.*, 1999]. This new definition of a stable model also turns out to be closely related to equilibrium logic [Pearce, 1997], a logic based on the concept of a Kripke-model in the logic of here-and-there. Also, we will show that many theorems about programs with nested expressions extend to arbitrary propositional theories.

On top of arbitrary propositional formulas, we give our definition of an aggregate. Our extension of the semantics to aggregates treats aggregates in a way similar to propositional connectives. Aggregates can be viewed either as primitive constructs or as abbreviations for propositional formulas; both approaches lead to the same concept of a stable model. The second view is important because it allows us to use theorems about stable models of propositional formulas in the presence of aggregates. As an example of application of such theorems, we use them to prove the correctness of an ASP program with aggregates that encodes a combinatorial auction problem.

Syntactically, our aggregates can occur in any part of a formula, even nested inside each other. (The idea of “nested aggregates” is not completely new, as the proof of Theorem 3(a) in [Ferraris, 2007] involves “nested weight constraints”.) In our definition of an aggregate we can have, in the same program/theory, many other kinds of constructs, such as choice rules and disjunction in the head, while other definitions allow only a subset of them. Our aggregates seem not to exhibit the unintuitive behaviours of other definitions of aggregates.

It also turns out that a minor syntactical modification of programs with FLP-

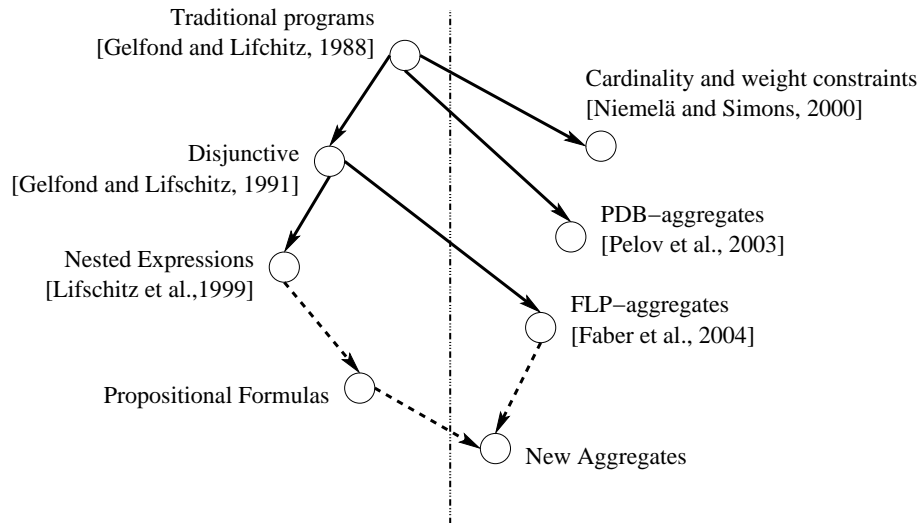


Figure 2: The proposed extensions

aggregates allows us to view them as a special kind of our aggregates. (The new picture of extensions is shown in Figure 2.) Consequently, we also have a “propositional” representation of FLP-aggregates. We use this fact to compare them with other aggregates that have a characterization in terms of nested expressions. (As we said, [Ferraris and Lifschitz, 2005b] showed that weight constraints can be expressed as nested expressions, and also [Pelov *et al.*, 2003] implicitly defined PDB-aggregates in terms of nested expressions.) We will show that all characterizations of aggregates are essentially equivalent to each other when the aggregates are monotone or antimonotone and without negation, while there are differences in the other cases.⁴

The paper is divided into three main parts. We start, in the next section, with the new definition of a stable model for propositional theories, their properties and comparisons with previous definitions of stable models and equilibrium logic. In Section 3 we present our aggregates, their properties and the comparisons with other definitions of aggregates. Section 4 contains all proofs for the theorems of this paper. The paper ends with the conclusions in Section 5.

Preliminary reports on some results of this paper were published in [Ferraris, 2005].

⁴The important role of monotonicity in aggregates has already been shown, for instance, in [Faber *et al.*, 2004].

2 Stable models of propositional theories

2.1 Definition

Usually, in logic programming, variables are allowed. As in most definitions of a stable model, we assume that the variables have been replaced by constants in a process called “grounding” (see, for instance, [Gelfond and Lifschitz, 1988]), so that we can consider the signature to be essentially propositional.

(Propositional) formulas are built from atoms and the 0-place connective \perp (false), using the connectives \wedge , \vee and \rightarrow . Even if our definition of a stable model below applies to formulas with all propositional connectives, we will consider \top as an abbreviation for $\perp \rightarrow \perp$, a formula $\neg F$ as an abbreviation for $F \rightarrow \perp$ and $F \leftrightarrow G$ as an abbreviation for $(F \rightarrow G) \wedge (G \rightarrow F)$. This will keep notation for other sections simpler. It can be shown that these abbreviations perfectly capture the meaning of \top , \neg and \leftrightarrow as primitive connectives in the stable model semantics.

A *(propositional) theory* is a set of formulas. As usual in logic programming, truth assignments will be viewed as sets of atoms; we will write $X \models F$ to express that a set X of atoms satisfies a formula F , and similarly for theories.

An implication $F \rightarrow G$ can be also written as a “rule” $G \leftarrow F$, so that traditional programs, disjunctive programs and programs with nested expressions (reviewed in Section 2.2) can be seen as special cases of propositional theories.⁵

We will now define when a set X of atoms is a stable model of a propositional theory Γ . For the rest of the section X denotes a set of atoms.

The *reduct* F^X of a propositional formula F relative to X is obtained from F by replacing each maximal subformula not satisfied by X with \perp . That is, recursively,

- $\perp^X = \perp$;
- for every atom a , if $X \models a$ then a^X is a , otherwise it is \perp ; and
- for every formulas F and G and any binary connective \otimes , if $X \models F \otimes G$ then $(F \otimes G)^X$ is $F^X \otimes G^X$, otherwise it is \perp .

This definition of reduct is similar to a transformation proposed in [Osorio *et al.*, 2004, Section 4.2].

⁵Traditionally, conjunction is represented in a logic program by a comma, disjunction by a semi-colon, and negation as failure as *not*.

For instance, if X contains p but not q then

$$\begin{aligned}
(p \leftarrow \neg q)^X &= (p \leftarrow (q \rightarrow \perp))^X = p \leftarrow (\perp \rightarrow \perp) = p \leftarrow \top \\
(q \leftarrow \neg p)^X &= (q \leftarrow (p \rightarrow \perp))^X = \perp \leftarrow \perp \\
((p \rightarrow q) \vee (q \rightarrow p))^X &= \perp \vee (\perp \rightarrow p)
\end{aligned} \tag{2}$$

The *reduct* Γ^X of a propositional theory Γ relative to X is $\{F^X : F \in \Gamma\}$. A set X of atoms is a *stable model* of Γ if X is a minimal set satisfying Γ^X .

For instance, let Γ be the theory consisting of

$$\begin{aligned}
p &\leftarrow \neg q \\
q &\leftarrow \neg p
\end{aligned} \tag{3}$$

Theory Γ is actually a traditional program, a logic program in the sense of [Gelfond and Lifschitz, 1988] (more details in the next section). Set $\{p\}$ is a stable model of Γ ; indeed, by looking at the first two lines of (2) we can see that $\Gamma^{\{p\}}$ is $\{p \leftarrow \top, \perp \leftarrow \perp\}$, which is satisfied by $\{p\}$ but not by its unique proper subset \emptyset . It is easy to verify that $\{q\}$ is the only other stable model of Γ . Similarly, it is not difficult to see that $\{p\}$ is the only stable model of the theory

$$\begin{aligned}
&(p \rightarrow q) \vee (q \rightarrow p) \\
&p
\end{aligned} \tag{4}$$

(The reduct relative to $\{p\}$ is $\{\perp \vee (\perp \rightarrow p), p\}$).

As the name suggests, a stable model of a propositional theory Γ is a model — in the sense of classical logic — of Γ . Indeed, it follows from the easily verifiable fact that, for each set X of atoms, $X \models \Gamma^X$ iff $X \models \Gamma$. On the other hand, formulas that are equivalent in classical logic may have different stable models: for instance, $\{\neg\neg p\}$ has no stable models, while $\{p\}$ has stable model $\{p\}$. Proposition 5 below will give some characterizations of transformations that preserves stable models. Notice that classically equivalent transformations can be applied to the reduct of a theory, as the sets of atoms that are minimal don't change.

Finally, a note about a second kind of negation in propositional theories. In [Ferraris and Lifschitz, 2005a, Section 3.9], atoms were divided into two groups: “positive” and “negative”, so that each negative atom has the form $\sim a$, where a is a positive atom. Symbol \sim is called “strong negation”, to distinguish it from the connective \neg , which is called *negation as failure*.⁶ In presence of strong negation,

⁶Strong negation was introduced in the syntax of logic programs in [Gelfond and Lifschitz, 1991]. In that paper, it was called “classical negation” and treated not as a part of an atom, but rather as a logical operator.

kind of rule	syntax
traditional	$a \leftarrow l_1 \wedge \cdots \wedge l_n$
disjunctive	$a_1 \vee \cdots \vee a_m \leftarrow l_1 \wedge \cdots \wedge l_n$
with nested expressions	$F \leftarrow G$ (F and G are nested expressions)

Figure 3: Syntax of “propositional” logic programs. Each a, a_1, \dots, a_m ($m \geq 0$) denotes an atom, and each l_1, \dots, l_n ($n \geq 0$) a literal — an atom possibly prefixed by \neg . A *nested expression* is any formula that contains no implications other than negations or \top .

the stable model semantics says that only sets of atoms that don’t contain both atoms a and $\sim a$ can be stable models. For simplicity, we will make no distinctions between positive and negative atoms, considering that we can remove the sets of atoms containing any pair of atoms a and b from the stable models of a theory by adding a formula $\neg(a \wedge b)$ to the theory. (See Proposition 7).

2.2 Relationship with previous definitions of a stable model

As mentioned in the introduction, a propositional theory is the extension of traditional programs [Gelfond and Lifschitz, 1988], disjunctive programs [Gelfond and Lifschitz, 1991] and programs with nested expressions [Lifschitz *et al.*, 1999] (see Figure 2). We want to compare the definition of a stable model from the previous section with the definitions in the three papers cited above.

The syntax of a *traditional rule*, *disjunctive rule* and *rule with nested expressions* are shown in Figure 3. We understand an empty conjunction as \top and an empty disjunction as \perp , so that traditional and disjunctive rules are also rules with nested expressions. The part before and after the arrow \leftarrow are called the *head* and the *body* of the rule, respectively. When the body is empty (or \top), we can denote the whole rule by its head. A *logic program* is a set of rules. If all rules in a logic program are traditional then we say that the program is *traditional* too, and similarly for the other two kinds of rules.

For instance, (3) is a traditional program as well as a disjunctive program and a program with nested expressions. On the other hand, (4) is not a logic program of any of those kinds, because of the first formula that contains implications nested in a disjunction.

For all kinds of programs described above, the definition of a stable model is similar to ours for propositional theories: to check whether a set X of atoms is a stable model of a program Π , we (i) compute the reduct of Π relative to X , and (ii) verify if X is a minimal model of such reduct. On the other hand, the way in which the reduct is computed is different. We consider the definition from [Lifschitz *et al.*,

1999], as the definitions from [Gelfond and Lifschitz, 1988, 1991] are essentially its special cases.

The *reduct* Π^X of a program Π with nested expressions relative to a set X of atoms is the result of replacing, in each rule of Π , each maximal subformula of the form $\neg F$ with \top if $X \models \neg F$, and with \perp otherwise. Set X is a *stable model* of Π if it is a minimal model of Π^X .⁷

For instance, if Π is (3) then the reduct $\Pi^{\{p\}}$ is

$$\begin{aligned} p &\leftarrow \top \\ q &\leftarrow \perp, \end{aligned}$$

while Π^\emptyset is

$$\begin{aligned} p &\leftarrow \top \\ q &\leftarrow \top, \end{aligned}$$

The stable models of Π — based on this definition of the reduct — are the same ones that we computed in the previous section using the newer definition of a reduct: $\{p\}$ and $\{q\}$. On the other hand, there are differences in the value of the reducts: for instance, we have just seen that Π^\emptyset is classically equivalent to $\{p, q\}$, while $\Pi^\emptyset = \{\perp, \perp\}$. However, some similarities between these definitions exist. For instance, negations are treated essentially in the same way: a nested expression $\neg F$ is transformed into \perp if $X \models F$, and into \top otherwise, under both definitions of a reduct.

The following proposition states a more general relationship between the new definition and the 1999 definition of a reduct.

Proposition 1. *For any program Π with nested expressions and any set X of atoms, Π^X is equivalent, in the sense of classical logic,*

- *to \perp , if $X \not\models \Pi$, and*
- *to the program obtained from Π^X by replacing all atoms that do not belong to X by \perp , otherwise.*

Corollary 1. *Given two sets of atoms X and Y with $Y \subseteq X$ and any program Π with nested expressions, $Y \models \Pi^X$ iff $X \models \Pi$ and $Y \models \Pi^X$.*

From the corollary above, one of the main claims of this paper follows, that our definition of a stable model is an extension of the definition for programs with nested expressions.

⁷We underline the set X in Π^X to distinguish this definition of a reduct from the one from the previous section.

Proposition 2. *For any program Π with nested expressions, the collections of stable models of Π according to our definition and according to [Lifschitz et al., 1999] are identical.*

2.3 Relationship with Equilibrium Logic

Equilibrium logic [Pearce, 1997, 1999] is defined in terms of Kripke models in the logic of here-and-there, a logic intermediate between intuitionistic and classical logic.

The logic of here-and-there is a 3-valued logic, where an interpretation (called an *HT-interpretation*) is represented by a pair (X, Y) of sets of atoms where $X \subseteq Y$. Intuitively, atoms in X are considered “true”, atoms not in Y are considered “false”, and all other atoms (that belong to Y but not X) are “undefined”.

An HT-interpretation (X, Y) *satisfies* a formula F (symbolically, $(X, Y) \models F$) based on the following recursive definition (a stands for an atom):

- $(X, Y) \models a$ iff $a \in X$,
- $(X, Y) \not\models \perp$,
- $(X, Y) \models F \wedge G$ iff $(X, Y) \models F$ and $(X, Y) \models G$,
- $(X, Y) \models F \vee G$ iff $(X, Y) \models F$ or $(X, Y) \models G$,
- $(X, Y) \models F \rightarrow G$ iff $(X, Y) \models F$ implies $(X, Y) \models G$, and Y satisfies $F \rightarrow G$ in classical logic.

An HT-interpretation (X, Y) *satisfies* a propositional theory if it satisfies all the elements of the theory. Two formulas are *equivalent* in the logic of here-and-there if they are satisfied by the same HT-interpretations.

Equilibrium logic defines when a set X of atoms is an equilibrium model of a propositional theory Γ . Set X is an *equilibrium model* of Γ if $(X, X) \models \Gamma$ and, for all proper subsets Z of X , $(Z, X) \not\models \Gamma$.

A relationship between the concept of a model in the logic of here-and-there, and satisfaction of the reduct exists.

Proposition 3. *For any formula F and any HT-interpretation (X, Y) , $(X, Y) \models F$ iff $X \models F^Y$.*

Next proposition compares the concept of an equilibrium model with the new definition of a stable model.

Proposition 4. *For any theory, its models in the sense of equilibrium logic are identical to its stable models.*

This proposition offers another way of proving Proposition 2, as [Lifschitz *et al.*, 2001] showed that the equilibrium models of a program with nested expressions are the stable models of the same program in the sense of [Lifschitz *et al.*, 1999].

2.4 Properties of propositional theories

This section shows how several theorems about logic programs with nested expressions can be extended to propositional theories.

2.4.1 Strong equivalence

Two theories Γ_1 and Γ_2 are *strongly equivalent* if, for every theory Γ , $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same stable models.

Proposition 5. *For any two theories Γ_1 and Γ_2 , the following conditions are equivalent:*

- (i) Γ_1 is strongly equivalent to Γ_2 ,
- (ii) Γ_1 is equivalent to Γ_2 in the logic of here-and-there, and
- (iii) for each set X of atoms, Γ_1^X is equivalent to Γ_2^X in classical logic.

The equivalence between (i) and (ii) is essentially Lemma 4 from [Lifschitz *et al.*, 2001] about equilibrium logic. The equivalence between (i) and (iii) is similar to Theorem 1 from [Turner, 2003] about nested expressions, but simpler and more general. Notice that (iii) cannot be replaced by

- (iii') for each set X of atoms, Γ_1^X is equivalent to Γ_2^X in classical logic,

not even when Γ_1 and Γ_2 are programs with nested expressions. Indeed, $\{p \leftarrow \neg p\}$ is strongly equivalent to $\{\perp \leftarrow \neg p\}$, but $\{p \leftarrow \neg p\}^\emptyset = \{p \leftarrow \top\}$ is not classically equivalent to $\{\perp \leftarrow \neg p\}^\emptyset = \{\perp \leftarrow \top\}$.

Replacing, in a theory Γ , a (sub)formula F with a formula G is guaranteed to preserve strong equivalence iff F is strongly equivalent to G . Indeed, strong equivalence between F and G is clearly a necessary condition: take $\Gamma = \{F\}$. It is also sufficient because — as in classical logic — replacements of formulas with equivalent formulas in the logic of here-and-there preserves equivalence in the same logic.

Cabalar and Ferraris [2007] showed that any propositional theory is strongly equivalent to a logic program with nested expressions. That is, a propositional theory can be seen as a different way of writing a logic program. This shows that the concept of a stable model for propositional theories is not too different from the concept of a stable model for a logic program.

2.4.2 Other properties

To state several propositions below, we need the following definitions. Recall that an expression of the form $\neg F$ is an abbreviation for $F \rightarrow \perp$, and equivalences are the conjunction of two opposite implications. An occurrence of an atom in a formula is *positive* if it is in the antecedent of an even number of implications. An occurrence is *strictly positive* if such number is 0, and *negative* if it odd.⁸ For instance, in a formula $(p \rightarrow r) \rightarrow q$, the occurrences of p and q are positive, the one of r is negative, and the one of q is strictly positive.

The following proposition is an extension of the property that in each stable model of a program, each atom occurs in the head of a rule of that program [Lifschitz, 1996, Section 3.1]. An atom is a *head atom* of a theory Γ if it has a strictly positive occurrence in Γ .⁹

Proposition 6. *Each stable model of a theory Γ consists of head atoms of Γ .*

A rule is called a *constraint* if its head is \perp . In a logic program, adding constraints to a program Π removes the stable models of Π that don't satisfy the constraints. A constraint can be seen as a formula of the form $\neg F$, a formula that doesn't have head atoms. Next proposition generalizes the property of logic programs stated above to propositional theories.

Proposition 7. *For every two propositional theories Γ_1 and Γ_2 such that Γ_2 has no head atoms, a set X of atoms is a stable model of $\Gamma_1 \cup \Gamma_2$ iff X is a stable model of Γ_1 and $X \models \Gamma_2$.*

The following two propositions are generalizations of propositions stated in [Ferraris and Lifschitz, 2005b] in the case of logic programs. We say that an occurrence of an atom is *in the scope of negation* when it occurs in a formula $\neg F$.

Proposition 8 (Lemma on Explicit Definitions). *Let Γ be any propositional theory, and Q a set of atoms not occurring in Γ . For each $q \in Q$, let $Def(q)$ be a formula that doesn't contain any atoms from Q . Then $X \mapsto X \setminus Q$ is a 1-1 correspondence between the stable models of $\Gamma \cup \{Def(q) \rightarrow q : q \in Q\}$ and the stable models of Γ .*

Proposition 9 (Completion Lemma). *Let Γ be any propositional theory, and Q a set of atoms that have positive occurrences in Γ only in the scope of negation. For each $q \in Q$, let $Def(q)$ be a formula such that all negative occurrences of atoms*

⁸The concept of a positive and negative occurrence of an atom should not be confused by the concept of a "positive" and "negative" atom mentioned at the end of Section 2.1.

⁹In case of programs with nested expressions, it is easy to check that head atoms are atoms that occur in the head of a rule outside the scope of negation \neg .

from Q in $Def(q)$ are in the scope of negation. Then $\Gamma \cup \{Def(q) \rightarrow q : q \in Q\}$ and $\Gamma \cup \{Def(q) \leftrightarrow q : q \in Q\}$ have the same stable models.

The following proposition is essentially a generalization of the splitting set theorem from [Lifschitz and Turner, 1994] and [Erdoğan and Lifschitz, 2004], which allows to break logic programs/propositional theories into parts and compute the stable models separately. A formulation of this theorem has also been stated in [Ferraris and Lifschitz, 2005a] in the special case of theories consisting of a single formula.

Proposition 10 (Splitting Set Theorem). *Let Γ_1 and Γ_2 be two theories such that no atom occurring in Γ_1 is a head atom of Γ_2 . Let S be a set of atoms containing all head atoms of Γ_1 but no head atoms of Γ_2 . A set X of atoms is a stable model of $\Gamma_1 \cup \Gamma_2$ iff $X \cap S$ is a stable model of Γ_1 and X is a stable model of $(X \cap S) \cup \Gamma_2$.*

2.5 Computational complexity

Since the concept of a stable model is equivalent to the concept of an equilibrium model, checking the existence of a stable model of a propositional theory is a Σ_2^P -complete problem as for equilibrium models [Pearce *et al.*, 2001]. Notice that the existence of a stable model of a disjunctive program is already Σ_2^P -hard [Eiter and Gottlob, 1993, Corollary 3.8].

The existence of a stable model for a traditional program is a NP-complete problem [Marek and Truszczyński, 1991]. The same holds, more generally, for logic programs with nested expressions where the head of each rule is an atom or \perp . (We call programs of this kind *nondisjunctive*). We may wonder if the same property holds for arbitrary sets of formulas of the form $F \rightarrow a$ and $F \rightarrow \perp$. The answer is negative: the following lemma shows that as soon as we allow implications in formulas F then we have the same expressivity — and then complexity — as disjunctive rules.

Lemma 1. *Rule*

$$l_1 \wedge \dots \wedge l_m \rightarrow a_1 \vee \dots \vee a_n$$

($n > 0, m \geq 0$) where a_1, \dots, a_n are atoms and l_1, \dots, l_m are literals, is strongly equivalent to the set of n implications ($i = 1, \dots, n$)

$$(l_1 \wedge \dots \wedge l_m \wedge (a_1 \rightarrow a_i) \wedge \dots \wedge (a_n \rightarrow a_i)) \rightarrow a_i. \quad (5)$$

Proposition 11. *The problem of the existence of a stable model of a theory consisting of formulas of the form $F \rightarrow a$ and $F \rightarrow \perp$ is Σ_2^P -complete.*

We will see, in Section 3.5, that the conjunctive terms in the antecedent of (5) can equivalently be replaced by aggregates of a simple kind, thus showing that allowing aggregates in nondisjunctive programs increases their computational complexity.

3 Aggregates

3.1 Syntax and semantics

A *formula with aggregates* is defined recursively as follows:

- atoms and \perp are formulas with aggregates¹⁰,
- propositional combinations of formulas with aggregates are formulas with aggregates, and
- any expression of the form

$$op\langle\{F_1 = w_1, \dots, F_n = w_n\}\rangle \prec N \quad (6)$$

where

- op is (a symbol for) a function from multisets of real numbers to $\mathcal{R} \cup \{-\infty, +\infty\}$ (such as sum, product, min, max, etc.),
- F_1, \dots, F_n are formulas with aggregates, and w_1, \dots, w_n are (symbols for) real numbers (“weights”),
- \prec is (a symbol for) a binary relation between real numbers, such as \leq and $=$, and
- N is (a symbol for) a real number,

is a formula with aggregates.

A *theory with aggregates* is a set of formulas with aggregates. A formula of the form (6) is called an *aggregate*.

The intuitive meaning of an aggregate is explained by the following clause, which extends the definition of satisfaction of propositional formulas to arbitrary formulas with aggregates. For any aggregate (6) and any set X of atoms, let W_X be the multiset W consisting of the weights w_i ($1 \leq i \leq n$) such that $X \models F_i$; we say that X *satisfies* (6) if $op(W_X) \prec N$. For instance,

$$sum\langle\{p = 1, q = 1\}\rangle \neq 1 \quad (7)$$

¹⁰Recall that \top is an abbreviation for $\perp \rightarrow \perp$

is satisfied by the sets of atoms that satisfy both p and q or none of them.

As usual, we say that X satisfies a theory Γ with aggregates if X satisfies all formulas in Γ . We extend the concept of classical equivalence to formulas/theories with aggregates.

We extend the definition of a stable models of propositional theories (Section 2) to cover aggregates, in a very natural way. Let X be a set of atoms. The *reduct* F^X of a formula F with aggregates relative to X is again the result of replacing each maximal formula not satisfied by X with \perp . That is, it is sufficient to add a clause relative to aggregates to the recursive definition of a reduct: for an aggregate A of the form (6),

$$A^X = \begin{cases} \text{op}\langle\{F_1^X = w_1, \dots, F_n^X = w_n\}\rangle \prec N, & \text{if } X \models A, \\ \perp, & \text{otherwise.} \end{cases}$$

This is similar to the clause for binary connectives:

$$(F \otimes G)^X = \begin{cases} F^X \otimes G^X, & \text{if } X \models F \otimes G, \\ \perp, & \text{otherwise.} \end{cases}$$

The rest of the definition of a stable model remains the same: the *reduct* Γ^X of a theory Γ with aggregates is $\{F^X : F \in \Gamma\}$, and X is a *stable model* of Γ if X is a minimal model of Γ^X .

Consider, for instance, the theory Γ consisting of one formula

$$\text{sum}\langle\{p = -1, q = 1\}\rangle \geq 0 \rightarrow q. \quad (8)$$

Set $\{q\}$ is a stable model of Γ . Indeed, since both the antecedent and consequent of (8) are satisfied by $\{q\}$, $\Gamma^{\{q\}}$ is

$$\text{sum}\langle\{\perp = -1, q = 1\}\rangle \geq 0 \rightarrow q.$$

The antecedent of the implication above is satisfied by every set of atoms, so the whole formula is equivalent to q . Consequently, $\{q\}$ is the minimal model of $\Gamma^{\{q\}}$, and then a stable model of Γ .

3.2 Aggregates as Propositional Formulas

A formula/theory with aggregates can also be seen as a normal propositional formula/theory, by identifying (6) with the formula

$$\bigwedge_{I \subseteq \{1, \dots, n\} : \text{op}\langle\{w_i : i \in I\}\rangle \not\prec N} \left(\left(\bigwedge_{i \in I} F_i \right) \rightarrow \left(\bigvee_{i \in \bar{I}} F_i \right) \right), \quad (9)$$

where \bar{I} stands for $\{1, \dots, n\} \setminus I$, and $\not\prec$ is the negation of \prec .

For instance, if we consider aggregate (7), the conjunctive terms in (9) correspond to the cases when the sum of weights is 1, that is, when $I = \{1\}$ and $I = \{2\}$. The two implications are $q \rightarrow p$ and $p \rightarrow q$ respectively, so that (7) is

$$(q \rightarrow p) \wedge (p \rightarrow q). \quad (10)$$

Similarly,

$$\text{sum}\langle\{p = 1, q = 1\}\rangle = 1 \quad (11)$$

is

$$(p \vee q) \wedge \neg(p \wedge q). \quad (12)$$

Even though (11) can be seen as the negation of (7), the negation of (12) is not strongly equivalent to (10) (although they are classically equivalent). This shows that it is generally incorrect to “move” a negation from a binary relation symbol (such as \neq) in front of the aggregate as the unary connective \neg , and vice versa.

Next proposition shows that this understanding of aggregates as propositional formulas is equivalent to the semantics for theories with aggregates of the previous section. Two formulas with aggregates are *classically equivalent* to each other if they are satisfied by the same sets of atoms.

Proposition 12. *Let A be an aggregate of the form (6) and let G be the corresponding formula (9). Then*

- (a) G is classically equivalent to A , and
- (b) for any set X of atoms, G^X is classically equivalent to A^X .

Treating aggregates as propositional formulas allows us to apply many properties of propositional theories presented in Section 2.4 to theories with aggregates also. Consequently, we have the concept of an head atom, of strong equivalence, we can use the completion lemma and so on. We will use several of those properties to prove Proposition 14 below. In the rest of the paper we will often make no distinctions between the two ways of defining the semantics of aggregates discussed here.

Notice that replacing, in a theory, an aggregate of the form (6) with a formula that is not strongly equivalent to the corresponding formula (9) may lead to different stable models. This shows that there is no other way (modulo strong equivalence) of representing our aggregates as propositional formulas.

3.3 Monotone Aggregates

An aggregate $op\langle\{F_1 = w_1, \dots, F_n = w_n\}\rangle \prec N$ is *monotone* if, for each pair of multisets W_1, W_2 such that $W_1 \subseteq W_2 \subseteq \{w_1, \dots, w_n\}$, $op(W_2) \prec N$ is true whenever $op(W_1) \prec N$ is true. The definition of an *antimonotone* aggregate is similar, with $W_1 \subseteq W_2$ replaced by $W_2 \subseteq W_1$.

For instance,

$$sum\langle\{p = 1, q = 1\}\rangle > 1 \quad (13)$$

is monotone, and

$$sum\langle\{p = 1, q = 1\}\rangle < 1. \quad (14)$$

is antimonotone. An example of an aggregate that is neither monotone nor antimonotone is (7).

Proposition 13. *For any aggregate $op\langle\{F_1 = w_1, \dots, F_n = w_n\}\rangle \prec N$, formula (9) is strongly equivalent to*

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\bigvee_{i \in \bar{I}} F_i \right) \quad (15)$$

if the aggregate is monotone, and to

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\neg \bigwedge_{i \in I} F_i \right) \quad (16)$$

if the aggregate is antimonotone.

In other words, if $op\langle S \rangle \prec N$ is monotone then the antecedents of the implications in (9) can be dropped. Similarly, in case of antimonotone aggregates, the consequents of these implications can be replaced by \perp . In both cases, (9) is turned into a nested expression, if F_1, \dots, F_n are nested expressions.

For instance, aggregate (13) is normally written as formula

$$(p \vee q) \wedge (p \rightarrow q) \wedge (q \rightarrow p).$$

Since the aggregate is monotone, it can also be written, by Proposition 13, as nested expression

$$(p \vee q) \wedge q \wedge p,$$

which is strongly equivalent to $q \wedge p$. Similarly, aggregate (14) is normally written as formula

$$((p \wedge q) \rightarrow \perp) \wedge (p \rightarrow q) \wedge (q \rightarrow p);$$

since the aggregate is nonmonotone, it can also be written as nested expression

$$\neg(p \wedge q) \wedge \neg p \wedge \neg q,$$

which is strongly equivalent to $\neg p \wedge \neg q$.

On the other hand, if an aggregate is neither monotone nor antimonotone, it may be not possible to find a nested expression strongly equivalent to (9), even if F_1, \dots, F_n are nested expressions. This is the case for (7). Indeed, the formula (9) corresponding to (7) is (10), whose reduct relative to $\{p, q\}$ is (10). Consequently, by Proposition 5, for any formula G strongly equivalent to (10), $G^{\{p,q\}}$ is classically equivalent to (10). On the other hand, the reduct of nested expressions are essentially AND-OR combinations of atoms, \top and \perp (negations either become \perp or \top in the reduct), and no formula of this kind is classically equivalent to (10).

In some uses of ASP, aggregates that are neither monotone nor antimonotone are essential, as discussed in the next section.

3.4 Example

We consider the following variation of the combinatorial auction problem [Baral and Uyan, 2001], which can be naturally formalized using an aggregate that is neither monotone nor antimonotone.

Joe wants to move to another town and has the problem of removing all his bulky furniture from his old place. He has received some bids: each bid may be for one piece or several pieces of furniture, and the amount offered can be negative (if the value of the pieces is lower than the cost of removing them). A junkyard will take any object not sold to bidders, for a price. The goal is to find a collection of bids for which Joe doesn't lose money, if there is any.

Assume that there are n bids, denoted by atoms b_1, \dots, b_n . We express by the formulas

$$b_i \vee \neg b_i \tag{17}$$

($1 \leq i \leq n$) that Joe is free to accept any bid or not. Clearly, Joe cannot accept two bids that involve the selling of the same piece of furniture. So, for every such pair i, j of bids, we include the formula

$$\neg(b_i \wedge b_j). \tag{18}$$

Next, we need to express which pieces of the furniture have not been given to bidders. If there are m objects we can express that an object i is sold by bid j by adding the rule

$$b_j \rightarrow s_i \tag{19}$$

to our theory.

Finally, we need to express that Joe doesn't lose money by selling his items. This is done by the aggregate

$$\text{sum}\langle\{b_1 = w_1, \dots, b_n = w_n, \neg s_1 = -c_1, \dots, \neg s_m = -c_m\}\rangle \geq 0, \quad (20)$$

where each w_i is the amount of money (possibly negative) obtained by accepting bid i , and each c_i is the money requested by the junkyard to remove item i . Note that (20) is neither monotone nor antimonotone.

We define a *solution* to Joe's problem as a set of accepted bids such that

- (a) the bids involve selling disjoint sets of items, and
- (b) the sum of the money earned from the bids is greater than the money spent giving away the remaining items.

Proposition 14. $X \mapsto \{i : b_i \in X\}$ is a 1-1 correspondence between the stable models of the theory consisting of formulas (17)–(20) and a solution to Joe's problem.

3.5 Computational Complexity

Since theories with aggregates generalize disjunctive programs, the problem of the existence of a stable model of a theory with aggregates clearly is Σ_2^P -hard.¹¹ We need to check in which class of the computational hierarchy this problem belongs.

Even if propositional formulas corresponding to aggregates can be exponentially larger than the original aggregate, it turns out that (by treating aggregates as primitive constructs) the computation is not harder than for propositional theories.

Proposition 15. *If, for every aggregate, computing $op(W) \prec N$ requires polynomial time then the existence of a stable model of a theory with aggregates is a Σ_2^P -complete problem.*

For a nondisjunctive program with nested expressions the existence of a stable model is NP-complete. If we allow nonnested aggregates in the body, for instance by allowing rules

$$A_1 \wedge \dots \wedge A_n \rightarrow a$$

(A_1, \dots, A_n are aggregates and a is an atom or \perp) then the complexity increases to Σ_2^P . This follows from Lemma 1, since, in (5), each formula l_i is the propositional

¹¹We are clearly assuming weight not to be arbitrary real numbers but to belong to a countable subset of real numbers, such as integers of floating point numbers.

	monotone/antimonotone aggregates	generic aggregates	anti-chain property
weight constraints	NP-complete	NP-complete	NO
PDB-aggregates	NP-complete	Σ_P^2 -complete	YES
FLP-aggregates	NP-complete	Σ_P^2 -complete	YES
our aggregates	NP-complete	Σ_P^2 -complete	NO

Figure 4: Properties of definitions of programs with aggregates, in the case in which the head of each rule is an atom. We limit the syntax of our aggregates to the syntax allowed by the other formalisms. The complexity is relative to the problem of the existence of a stable model. The anti-chain property holds when no stable model can be a subset of another one.

representation of $sum\langle\{l_i = 1\}\rangle \geq 1$; similarly, each $a_j \rightarrow a_i$ is the propositional representation of $sum\langle\{a_j = -1, a_i = 1\}\rangle \geq 0$.

However, if we allow monotone and antimonotone aggregates only — even nested — in the antecedent, we are in class NP.

Proposition 16. *Consider theories with aggregates consisting of formulas of the form*

$$F \rightarrow a,$$

where a is an atom or \perp , and F contains monotone and antimonotone aggregates only, no equivalences and no implications other than negations. If, for every aggregate, computing $op(W) \prec N$ requires polynomial time then the problem of the existence of a stable model of theories of this kind is an NP-complete problem.

Similar results have been independently proven in [Calimeri *et al.*, 2005] for FLP-aggregates.

3.6 Other Formalisms

Figure 4 already shows that there are several differences between the various definitions of an aggregate. We analyze that more in details in the rest of this section.

3.6.1 Programs with weight constraints

Weight constraints are aggregates defined in [Niemelä and Simons, 2000] and implemented in answer set solver SMODELS. We simplify the syntax of weight constraints and of programs with weight constraints for clarity, without reducing its semantical expressivity.

Weight constraints are expressions of the form

$$N \leq \{l_1 = w_1, \dots, l_m = w_m\} \quad (21)$$

and

$$\{l_1 = w_1, \dots, l_m = w_m\} \leq N \quad (22)$$

where

- N is (a symbol for) a real number,
- each of l_1, \dots, l_m is a (symbol for) a literal, and w_1, \dots, w_m are (symbols for) real numbers.

An example of a weight constraint is (1).

The intuitive meaning of (21) is that the sum of the weights w_i for all the l_i that are true is not lower than N . For (22) the sum of weights is not greater than N . Often, $N_1 \leq S$ and $S \leq N_2$ are written together as $N_1 \leq S \leq N_2$. If a weight w is 1 then the part “= w ” is generally omitted. If all weights are 1 then a weight constraint is called a *cardinality constraint*.

A *rule with weight constraints* is an expression of the form

$$a \leftarrow C_1 \wedge \dots \wedge C_n \quad (23)$$

where a is an atom or \perp , and C_1, \dots, C_n ($n \geq 0$) are weight constraints.

Finally, a *program with weight constraints* is a set of rules with weight constraints. *Rules/programs with cardinality constraints* are rules/programs with weight constraints containing cardinality constraints only.

Programs with cardinality/weight constraints can be seen as a generalization of traditional programs, by identifying each literal l in the body of each rule with cardinality constraint $1 \leq \{l\}$.

The definition of a stable model from [Niemelä and Simons, 2000] requires first the elimination of negative weights from weight constraints. This is done by replacing each term $l_i = w_i$ where w_i is negative with $\bar{l}_i = -w_i$ (\bar{l}_i is the literal complementary to l_i) and increasing the bound by $-w_i$. For instance,

$$0 \leq \{p = 2, q = -1\}$$

is rewritten as

$$1 \leq \{p = 2, \neg q = 1\}.$$

Then [Niemelä and Simons, 2000] proposes a definition of a reduct and of a stable model for programs with weight constraints without negative weights. For

this paper, we prefer showing a translational, equivalent semantics of such programs from [Ferraris and Lifschitz, 2005b], that consists in replacing each weight constraint C with a nested expression $[C]$, preserving the stable models of the program: if C is (21) then $[C]$ is $(I \subseteq \{1, \dots, n\})$

$$\bigvee_{I : N \leq \sum_{i \in I} w_i} \left(\bigwedge_{i \in I} l_i \right) \quad (24)$$

and if C is (22) then $[C]$ is

$$\neg \bigvee_{I : N < \sum_{i \in I} w_i} \left(\bigwedge_{i \in I} l_i \right). \quad (25)$$

It turns out that the way of understanding a weight constraint C of this paper is not different from $[C]$ when all weights are nonnegative.

Proposition 17. *In presence of nonnegative weights only, $[N \leq S]$ is strongly equivalent to $\text{sum}\langle S \rangle \geq N$, and $[S \leq N]$ is strongly equivalent to $\text{sum}\langle S \rangle \leq N$.*

From this proposition, Propositions 2 and 5 of this paper, and Theorem 1 from [Ferraris and Lifschitz, 2005b] it follows that our concept of an aggregate captures the concept of weight constraints defined in [Niemelä and Simons, 2000] when all weights are nonnegative. It also captures the absence of the anti-chain property of its stable models: for instance,

$$p \leftarrow \{-p\} \leq 0$$

has stable models \emptyset and $\{p\}$ in both formalisms.

When we consider negative weights, however, such correspondence doesn't hold. For instance,

$$p \leftarrow 0 \leq \{p = 2, p = -1\}, \quad (26)$$

according to [Niemelä and Simons, 2000], has no stable models, while

$$p \leftarrow \text{sum}\langle \{p = 2, p = -1\} \rangle \geq 0 \quad (27)$$

has stable model \emptyset . An explanation of this difference can be seen in the pre-processing proposed by [Niemelä and Simons, 2000] that eliminates negative weights. For us, weight constraint $0 \leq \{p = 2, p = -1\}$, and the result $1 \leq \{p = 2, \neg p = 1\}$ of eliminating its negative weight, are semantically different.¹² Surprisingly,

¹²The fact that the process of eliminating negative weights is somehow unintuitive was already mentioned in [Ferraris and Lifschitz, 2005b] with the same example proposed in this section.

under the semantics of [Niemelä and Simons, 2000], $0 \leq \{p = 2, p = -1\}$ is different from $0 \leq \{p = 1\}$. In fact,

$$p \leftarrow 0 \leq \{p = 1\} \quad (28)$$

has stable model \emptyset , the same of (27), while (26) has none. Notice that summing weights that are all positive or all negative preserves stable models under both semantics.

The preliminary step of removing negative weights can be seen as a way of making weight constraints either monotone or antimonotone. This keeps the problem of the existence of a stable model in class NP, while we have seen in Section 3.5 that, under our semantics, even simple aggregates with the same intuitive meaning of $0 \leq \{p = 1, q = -1\}$ bring the same problem to class Σ_2^P .

3.6.2 PDB-aggregates

A *PDB-aggregate* is an expression of the form (6), where F_1, \dots, F_n are literals. A *program with PDB-aggregates* is a set of rules of the form

$$a \leftarrow A_1 \wedge \dots \wedge A_m,$$

where $m \geq 0$, a is an atom and A_1, \dots, A_m are PDB-aggregates.

As in the case of programs with weight constraints, a program with PDB-aggregates is a generalization of a traditional program, by identifying each literal l in the bodies of traditional programs by aggregate $sum\langle\{l = 1\}\rangle \geq 1$.

The semantics of [Pelov *et al.*, 2003] for programs with PDB-aggregates is based on a procedure that transforms programs with such aggregates into traditional programs.¹³ The procedure can be seen consisting of two parts. The first one essentially consists in rewriting each aggregate as a nested expression.¹⁴ The second part “unfolds” each rule into a strongly equivalent set of traditional rules. For our comparisons, only the first part is needed: each PDB-aggregate A of the form

$$op\langle\{l_1 = w_1, \dots, l_n = w_n\}\rangle \prec N$$

is replaced by the following nested expression A_{tr}

$$\bigvee_{I_1, I_2: I_1 \subseteq I_2 \subseteq \{1, \dots, n\} \text{ and for all } I \text{ such that } I_1 \subseteq I \subseteq I_2, op(W_I) \prec N} G_{(I_1, I_2)}$$

¹³A semantics for such aggregates was proposed in [Denecker *et al.*, 2001], based on the approximation theory [Denecker *et al.*, 2002]. But the first characterization of PDB-aggregates in terms of stable models is from [Pelov *et al.*, 2003]. [Son *et al.*, 2007] independently proposed a similar semantics.

¹⁴[Pelov *et al.*, 2003] doesn’t explicitly mention nested expressions.

where W_I stands for the multiset $\{w_i : i \in I\}$, and $G_{(I_1, I_2)}$ stands for

$$\bigwedge_{i \in I_1} l_i, \quad \bigwedge_{i \in \{1, \dots, n\} \setminus I_2} \bar{l}_i.$$

For instance, for the PDB-aggregate $A = \text{sum}\langle\{p = -1, q = 1\}\rangle \geq 0$, if we take $F_1 = p$, $F_2 = q$ then the pairs (I_1, I_2) that “contribute” to the disjunction in A_{tr} are

$$(\emptyset, \emptyset) \quad (\{2\}, \{2\}) \quad (\{1, 2\}, \{1, 2\}) \quad (\emptyset, \{2\}) \quad (\{2\}, \{1, 2\}).$$

The corresponding nested expressions $G_{(I_1, I_2)}$ are

$$\neg p \wedge \neg q \quad q \wedge \neg p \quad p \wedge q \quad \neg p \quad q.$$

It can be shown, using strong equivalent transformations (see Proposition 5) that the disjunction of such nested expressions can be rewritten as $\neg p \vee q$.

In case of monotone and antimonotone PDB-aggregates and in the absence of negation as failure, the semantics of Pelov *et al.* is equivalent to ours.

Proposition 18. *For any monotone or antimonotone PDB-aggregates A of the form (6) where F_1, \dots, F_n are atoms, A_{tr} is strongly equivalent to (9).*

The claim above is generally not true when either the aggregates are not monotone or antimonotone, or when some formula in the aggregate is a negative literal. Relatively to aggregates that are neither monotone nor antimonotone, the semantics of [Pelov *et al.*, 2003] seems to have the same unintuitive behaviour of [Niemelä and Simons, 2000]: for instance, according to [Pelov *et al.*, 2003], (27) has no stable models while

$$p \leftarrow \text{sum}\langle\{p = 1\}\rangle \geq 0$$

has stable model $\{p\}$.

To illustrate the problem with negative literals, consider the following Π :

$$\begin{aligned} p &\leftarrow \text{sum}\langle\{q = 1\}\rangle < 1 \\ q &\leftarrow \neg p \end{aligned} \tag{29}$$

and Π' :

$$\begin{aligned} p &\leftarrow \text{sum}\langle\{\neg p = 1\}\rangle < 1 \\ q &\leftarrow \neg p \end{aligned} \tag{30}$$

Intuitively, the two programs should have the same stable models. Indeed, the operation of replacing q with $\neg p$ in the first rule of Π should not affect the stable models since the second rule “defines” q as $\neg p$: it is the only rule with q in the head.

However, under the semantics of [Pelov *et al.*, 2003], Π has stable model $\{p\}$ only and Π' has stable model $\{q\}$ also. Under our semantics, both (29) and (30) have stable models $\{p\}$ and $\{q\}$.

Note that already the first rule of (30) has different stable models under the two semantics. Under ours, they are \emptyset and $\{p\}$. According to [Pelov *et al.*, 2003], only the empty set is a stable model; it couldn't have both stable models because stable models as defined in [Pelov *et al.*, 2003] have the anti-chain property.

3.6.3 FLP-aggregates

An *FLP-aggregate* is an expression of the form (6) where each of F_1, \dots, F_n is a conjunction of literals. A *program with FLP-aggregates* is a set of rules of the form

$$a_1 \vee \dots \vee a_n \leftarrow A_1 \wedge \dots \wedge A_m \wedge \neg A_{m+1} \wedge \dots \wedge \neg A_p \quad (31)$$

where $n \geq 0, 0 \leq m \leq p$, a_1, \dots, a_n are atoms and A_1, \dots, A_p are FLP-aggregates.

A program with FLP-aggregates is a generalization of a disjunctive program, by identifying each atom a in the bodies of disjunctive rules by aggregate $sum\langle\{a = 1\}\rangle \geq 1$.

The semantics of [Faber *et al.*, 2004] defines when a set of atoms is a stable model for a program with FLP-aggregates. The definition of satisfaction of an aggregate is identical to ours. The reduct, however, is computed differently. The reduct Π^X of a program Π with FLP-aggregates relative to a set X of atoms consists of the rules of the form (31) such that X satisfies its body. Set X is a stable model for Π if X is a minimal set satisfying Π^X .

For instance, let Π be the FLP-program

$$p \leftarrow sum\langle\{p = 2\}\rangle \geq 1.$$

The only stable model of Π is the empty set. Indeed, since the empty set doesn't satisfy the aggregate, $\Pi^\emptyset = \emptyset$, which has \emptyset as the unique minimal model; we can conclude that \emptyset is a stable model of Π . On the other hand, $\Pi^{\{p\}} = \Pi$ because $\{p\}$ satisfies the aggregate in Π . Since $\emptyset \models \Pi$, $\{p\}$ is not a minimal model of $\Pi^{\{p\}}$ and then it is not a stable model of Π .

This definition of a reduct is different from all other definitions of a reduct described in this paper (and also from many other definitions), in the sense that it may leave negation \neg in the body of a rule. For instance, the reduct of $a \leftarrow \neg b$ relative to $\{a\}$ is according to those definitions the fact a . In the theory of FLP-aggregates, the reduct doesn't modify the rule. On the other hand, this definition of a stable

model is equivalent to the definition of a stable model in the sense of [Gelfond and Lifschitz, 1991] (and successive definitions) when applied to disjunctive programs.

Next proposition shows a relationship between our concept of an aggregate and FLP-aggregates. An FLP-program is *positive* if, in each formula (31), $p = m$.

Next proposition shows that our semantics of aggregates is essentially an extension of the

Proposition 19. *The stable models of a positive FLP-program under our semantics are identical to its stable models in the sense of [Faber et al., 2004].*

The proposition doesn't apply to arbitrary FLP-aggregates as negation has different meanings in the two semantics. In case of [Faber et al., 2004], $\neg(op\langle S \prec N \rangle)$ is essentially the same as $op\langle S \not\prec N \rangle$, while we have seen, in Section 3.2, that this fact doesn't always hold in our semantics. The difference in meaning can be seen in the following example. Program

$$\begin{aligned} p &\leftarrow \neg q \\ q &\leftarrow \text{sum}\langle\{p = 1\}\rangle \leq 0 \end{aligned} \tag{32}$$

has two stable models $\{p\}$ and $\{q\}$ according to both semantics. However, if we replace q in the first rule with the body of the second (q is “defined” as $\text{sum}\langle\{p = 1\}\rangle \leq 0$ by the second rule), we get program

$$\begin{aligned} p &\leftarrow \neg(\text{sum}\langle\{p = 1\}\rangle \leq 0) \\ q &\leftarrow \text{sum}\langle\{p = 1\}\rangle \leq 0, \end{aligned} \tag{33}$$

which, according to [Faber et al., 2004], has only stable model $\{q\}$. We find it unintuitive.

It is the first rule of (33) that has a different meaning in the two semantics. The rule alone has different stable models: according to [Faber et al., 2004], its only stable models is \emptyset . Under our semantics, the stable models are \emptyset and $\{p\}$. As they don't have the anti-chain property, there is no program with FLP-aggregates that has such stable models under [Faber et al., 2004].

As a program with FLP-aggregate can be easily rewritten as a positive program with FLP-aggregate, our definition of an aggregate essentially generalizes the one of [Faber et al., 2004].

4 Proofs

4.1 Proofs of Propositions 3 and 4

Lemma 2. *For any formulas F_1, \dots, F_n ($n \geq 0$), any set X of atoms, and any connective $\otimes \in \{\vee, \wedge\}$, $(F_1 \otimes \dots \otimes F_n)^X$ is classically equivalent to $F_1^X \otimes \dots \otimes F_n^X$.*

F_n^X .

Proof. **Case 1:** $X \models F_1 \wedge \dots \wedge F_n$. Then, by the definition of reduct, $(F_1 \wedge \dots \wedge F_n)^X = F_1^X \wedge \dots \wedge F_n^X$. **Case 2:** $X \not\models F_1 \wedge \dots \wedge F_n$. Then $(F_1 \otimes \dots \otimes F_n)^X = \perp$; moreover, one of F_1, \dots, F_n is not satisfied by X , so that one of F_1^X, \dots, F_n^X is \perp . The case of disjunction is similar. \square

Proposition 3. For any formula F and any HT-interpretation (X, Y) , $(X, Y) \models F$ iff $X \models F^Y$.

Proof. It is sufficient to consider the case when Γ is a singleton $\{F\}$, where F contains only connectives $\wedge, \vee, \rightarrow$ and \perp . The proof is by structural induction on F .

- F is \perp . $X \not\models \perp$ and $(X, Y) \not\models \perp$.
- F is an atom a . $X \models a^Y$ iff $Y \models a$ and $X \models a$. Since $X \subseteq Y$, this means iff $X \models a$, which is the condition for which $(X, Y) \models a$.
- F has the form $G \wedge H$. $X \models (G \wedge H)^Y$ iff $X \models G^Y \wedge H^Y$ by Lemma 2, and then iff $X \models G^Y$ and $X \models H^Y$. This is equivalent, by induction hypothesis, to say that $(X, Y) \models G$ and $(X, Y) \models H$, and then that $(X, Y) \models G \wedge H$.
- The proof for disjunction is similar to the proof for conjunction.
- F has the form $G \rightarrow H$. $X \models (G \rightarrow H)^Y$ iff $X \models G^Y \rightarrow H^Y$ and $Y \models G \rightarrow H$, and then iff

$$X \models G^Y \text{ implies } X \models H^Y, \text{ and } Y \models G \rightarrow H.$$

This is equivalent, by the induction hypothesis, to

$$(X, Y) \models G \text{ implies } (X, Y) \models H, \text{ and } Y \models G \rightarrow H,$$

which is the definition of $(X, Y) \models G \rightarrow H$. \square

Proposition 4. For any theory, its models in the sense of equilibrium logic are identical to its stable models.

Proof. A set Y of atoms is an equilibrium model of Γ iff

$$(Y, Y) \models \Gamma \text{ and, for all proper subsets } X \text{ of } Y, (X, Y) \not\models \Gamma.$$

In view of Proposition 3, this is equivalent to the condition

$$Y \models \Gamma^Y \text{ and, for all proper subsets } X \text{ of } Y, X \not\models \Gamma^Y.$$

which means that Y is a stable model of Γ . \square

4.2 Proof of Propositions 1 and 2

We first need the recursive definition of reduct for programs with nested expressions from [Lifschitz *et al.*, 1999]. The reduct F^X of a nested expression F relative to a set X of atoms, as follows:

- $a^X = a$, $\perp^X = \perp$ and $\top^X = \top$,
- $(F \wedge G)^X = F^X \wedge G^X$ and $(F \vee G)^X = F^X \vee G^X$,
- $(\neg F)^X = \begin{cases} \perp, & \text{if } X \models F, \\ \top, & \text{otherwise,} \end{cases}$

Then the reduct $(F \leftarrow G)^X$ of a rule $F \leftarrow G$ with with nested expression is defined as $F^X \leftarrow G^X$, and the reduct Π^X of a program with nested expressions as the union of the reduct of its rules.

Lemma 3. *The reduct F^X of a nested expression F is equivalent, in the sense of classical logic, to the nested expression obtained from F^X by replacing all atoms that do not belong to X by \perp .*

Proof. The proof is by structural induction on F .

- When F is \perp or \top then $F^X = F = F^X$.
- For an atom a , $a^X = a$. The claim is immediate.
- Let F be a negation $\neg G$. If $X \models G$ then $F^X = \perp = F^X$; otherwise, $F^X = \neg\perp = \top = F^X$.
- for $F = G \otimes H$ ($\otimes \in \{\vee, \wedge\}$), F^X is $G^X \otimes H^X$, and, by Lemma 2, F^X is equivalent to $G^X \otimes H^X$. The claim now follows by the induction hypothesis.

□

Proposition 1. *For any program Π with nested expressions and any set X of atoms, Π^X is equivalent, in the sense of classical logic,*

- to \perp , if $X \not\models \Pi$, and
- to the program obtained from Π^X by replacing all atoms that do not belong to X by \perp , otherwise.

Proof. If $X \not\models \Pi$ then clearly Π^X contains \perp . Otherwise, Π^X consists of formulas $F^X \rightarrow G^X$ for each rule $G \leftarrow F \in \Pi$, and consequently for each rule $G^X \leftarrow F^X \in \Pi^X$. Since each F and G is a nested expression, the claim is immediate by Lemma 3. □

Proposition 2. *For any program Π with nested expressions, the collection of stable models of Π according to our definition and according to [Lifschitz et al., 1999] are identical.*

Proof. If $X \not\models \Pi$ then clearly Π^X contains \perp , and also $X \not\models \Pi^X$ (a well-known property about programs with nested expressions), so X is not a stable model under either definitions. Otherwise, by Corollary 1, the two reducts are satisfied by the same subsets of X . Then X is a minimal set satisfying Π^X iff it is a minimal set satisfying Π , and, by the definitions of a stable models X is a stable model of Π either for both definitions or for none of them. \square

4.3 Proofs of Propositions 5–7

Proposition 5. *For any two theories Γ_1 and Γ_2 , the following conditions are equivalent:*

- (i) Γ_1 is strongly equivalent to Γ_2 ,
- (ii) Γ_1 is equivalent to Γ_2 in the logic of here-and-there, and
- (iii) for each set X of atoms, Γ_1^X is equivalent to Γ_2^X in classical logic.

Proof. We will prove the equivalence between (i) and (ii) and between (ii) and (iii). We start with the former. Lemma 4 from [Lifschitz et al., 2001] tells that, for any two theories, the following conditions are equivalent:

- (a) for every theory Γ , theories $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same equilibrium models, and
- (b) Γ_1 is equivalent to Γ_2 in the logic of here-and-there.

Condition (b) is identical to (ii). Condition (a) can be rewritten, by Proposition 4, as

(a') for every theory Γ , theories $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same stable models, which means that Γ_1 is strongly equivalent to Γ_2 .

It remains to prove the equivalence between (ii) and (iii). Theory Γ_1 is equivalent to Γ_2 in the logic of here-and-there iff, for every set Y of atoms, the following condition holds:

$$\text{for every } X \subseteq Y, (X, Y) \models \Gamma_1 \text{ iff } (X, Y) \models \Gamma_2.$$

This condition is equivalent, by Proposition 3, to

$$\text{for every } X \subseteq Y, X \models \Gamma_1^Y \text{ iff } X \models \Gamma_2^Y.$$

Since Γ_1^Y and Γ_2^Y contain atoms from Y only (the other atoms are replaced by \perp in the reduct), this last condition expresses equivalence between Γ_1^Y and Γ_2^Y . \square

Lemma 4. *For any theory Γ , let S be a set of atoms that contains all head atoms of Γ . For any set X of atoms, if $X \models \Gamma$ then $X \cap S \models \Gamma^X$.*

Proof. It is clearly sufficient to prove the claim for Γ that is a singleton $\{F\}$. The proof is by induction on F .

- If $F = \perp$ then $X \not\models F$, and the claim is trivial.
- For an atom a , if $X \models a$ then $a^X = a$, but also $a \in S$, so that $X \cap S \models a^X$.
- If $X \models G \wedge H$ then $X \models G$ and $X \models H$. Consequently, by induction hypothesis, $X \cap S \models G^X$ and $X \cap S \models H^X$. It remains to notice that $(G \wedge H)^X = G^X \wedge H^X$.
- The case of disjunction is similar to the case of conjunction.
- If $X \models G \rightarrow H$ then $(G \rightarrow H)^X = G^X \rightarrow H^X$. Assume that $X \cap S \not\models G^X$. Consequently $G^X \neq \perp$ and then $X \models G$. It follows that, since $X \models G \rightarrow H$, $X \models H$. Since S contains all head atoms of H , the claim follows by the induction hypothesis.

\square

Lemma 5. *For any theory Γ and any set X of atoms, $X \models \Gamma^X$ iff $X \models \Gamma$.*

Proof. Reduct Γ^X is obtained from Γ by replacing some subformulas that are not satisfied by X with \perp . \square

Proposition 6. *Each stable model of a theory Γ consists of head atoms of Γ .*

Proof. Consider any theory Γ , the set S of head atoms of Γ , and a stable model X of Γ . By Lemma 5, $X \models \Gamma$, so that, by Lemma 4, $X \cap S \models \Gamma^X$. Since $X \cap S \subseteq X$ and no proper subset of X satisfies Γ^X , it follows that $X \cap S = X$, and consequently that $X \subseteq S$. \square

Proposition 7. *For every two propositional theories Γ_1 and Γ_2 such that Γ_2 has no head atoms, a set X of atoms is a stable model of $\Gamma_1 \cup \Gamma_2$ iff X is a stable model of Γ_1 and $X \models \Gamma_2$.*

Proof. If $X \models \Gamma_2$ then Γ_2^X is satisfied by every subset of X by Lemma 4, so that $(\Gamma_1 \cup \Gamma_2)^X$ is classically equivalent to Γ_1^X ; then clearly X is a stable model of $\Gamma_1 \cup \Gamma_2$ iff it is a stable model of Γ_1 . Otherwise, Γ_2^X contains \perp , and X cannot be a stable model of $\Gamma_1 \cup \Gamma_2$. \square

4.4 Proofs of Propositions 8 and 10

We start with the proof of Proposition 10. Some lemmas are needed.

Lemma 6. *If X is a stable model of Γ then Γ^X is equivalent to X .*

Proof. Since all atoms that occur in Γ^X belong to X , it is sufficient to show that the formulas are satisfied by the same subsets of X . By the definition of a stable model, the only subset of X satisfying Γ^X is X . \square

Lemma 7. *Let S be a set of atoms that contains all atoms that occur in a theory Γ_1 but does not contain any head atoms of a theory Γ_2 . For any set X of atoms, if X is a stable model of $\Gamma_1 \cup \Gamma_2$ then $X \cap S$ is a stable model of Γ_1 .*

Proof. Since X is a stable model of $\Gamma_1 \cup \Gamma_2$, $X \models \Gamma_1$, so that $X \cap S \models \Gamma_1$, and, by Lemma 5, $X \cap S \models \Gamma_1^{X \cap S}$. It remains to show that no proper subset Y of $X \cap S$ satisfies $\Gamma_1^{X \cap S}$. Let S' be the set of head atoms of Γ_2 , and let Z be $X \cap (S' \cup Y)$. We will show that Z has the following properties:

- (i) $Z \cap S = Y$;
- (ii) $Z \subset X$;
- (iii) $Z \models \Gamma_2^X$.

To prove (i), note that since S' is disjoint from S , and Y is a subset of $X \cap S$,

$$Z \cap S = X \cap (S' \cup Y) \cap S = X \cap Y \cap S = (X \cap S) \cap Y = Y.$$

To prove (ii), note that set Z is clearly a subset of X . It cannot be equal to X , because otherwise we would have, by (i),

$$Y = Z \cap S = X \cap S;$$

this is impossible, because Y is a proper subset of $X \cap S$. Property (iii) follows from Lemma 4, because $X \models \Gamma_2$, and $S' \cup Y$ contains all head atoms of Γ_2 .

Since X is a stable model of $\Gamma_1 \cup \Gamma_2$, from property (ii) we can conclude that $Z \not\models (\Gamma_1 \cup \Gamma_2)^X$. Consequently, by property (iii), $Z \not\models \Gamma_1^X$. Since all atoms that occur in Γ_1 belong to S , $\Gamma_1^X = \Gamma_1^{X \cap S}$, so that $Z \not\models \Gamma_1^{X \cap S}$. Since all atoms that occur in $\Gamma_1^{X \cap S}$ belong to S , it follows that $Z \cap S \not\models \Gamma_1^{X \cap S}$. By property (i), we conclude that $Y \not\models \Gamma_1^{X \cap S}$. \square

Proposition 10 (Splitting Set Theorem). *Let Γ_1 and Γ_2 be two theories such that no atom occurring in Γ_1 is a head atom of Γ_2 . Let S be a set of atoms containing all head atoms of Γ_1 but no head atoms of Γ_2 . A set X of atoms is a stable model of $\Gamma_1 \cup \Gamma_2$ iff $X \cap S$ is a stable model of Γ_1 and X is a stable model of $(X \cap S) \cup \Gamma_2$.*

Proof. We first prove the claim in the case when S contains all atoms of Γ_1 . If $X \cap S$ is not a stable model of Γ_1 then X is not a stable model of $\Gamma_1 \cup \Gamma_2$ by Lemma 7. Now suppose that $X \cap S$ is a stable model of Γ_1 . Then, by Lemma 6, $\Gamma_1^{X \cap S}$ is equivalent to $X \cap S$. Consequently,

$$\begin{aligned} (\Gamma_1 \cup \Gamma_2)^X &= \Gamma_1^X \cup \Gamma_2^X = \Gamma_1^{X \cap S} \cup \Gamma_2^X \leftrightarrow (X \cap S) \cup \Gamma_2^X \\ &= (X \cap S)^X \cup \Gamma_2^X = ((X \cap S) \cup \Gamma_2)^X \end{aligned}$$

We can conclude that X is a stable model of $\Gamma_1 \cup \Gamma_2$ iff X is a stable model of $\Gamma_2 \cup (X \cap S)$.

The most general case remains. Let S_1 be the set of all atoms in Γ_1 (the value of S for which we have already proved the claim). In view of the special case described above, it is sufficient to show that, for any set S of atoms that respects the hypothesis conditions,

$$X \cap S_1 \text{ is a stable model of } \Gamma_1 \text{ and } X \text{ is a stable model of } (X \cap S_1) \cup \Gamma_2 \quad (34)$$

holds iff

$$X \cap S \text{ is a stable model of } \Gamma_1 \text{ and } X \text{ is a stable model of } (X \cap S) \cup \Gamma_2. \quad (35)$$

Assume (34). Sets S and S_1 differ only for sets of atoms that are not head atoms of Γ_1 . Consequently, since $X \cap S_1$ is a stable model of Γ_1 , it follows from Proposition 6 that $X \cap S_1 = X \cap S$. We can then conclude that (35) follows from (34). The proof in the opposite direction is similar. \square

Lemma 8. *Let Γ be a theory, and let Y and Z be two disjoint sets of atoms such that no atom of Z is an head atoms of Γ . Let Γ' a theory obtained from Γ by replacing occurrences of atoms of Y with \top and occurrences of atoms of Z with \perp . Then $\Gamma \cup Y$ and $\Gamma' \cup Y$ have the same stable models.*

Proof. Atoms of Z are not head atoms of $\Gamma \cup Y$. Consequently, by Proposition 6, every stable model of $\Gamma \cup Y$ is disjoint from Z . It follows, by Proposition 7, that $\Gamma \cup Y$ has the same stable models of

$$\Gamma \cup Y \cup \{\neg a : a \in Z\}.$$

Similarly, $\Gamma' \cup Y$ has the same stable models of

$$\Gamma' \cup Y \cup \{\neg a : a \in Z\}.$$

It is a known property that the two theories above are equivalent to each other in intuitionistic logic, and then in the logic-of-here-and-there. Consequently, by Proposition 5, they are strongly equivalent to each other, and we can conclude that they have the same stable models. \square

Proposition 8. *Let Γ be any propositional theory, and Q a set of atoms not occurring in Γ . For each $q \in Q$, let $Def(q)$ be a formula that doesn't contain any atoms from Q . Then $X \mapsto X \setminus Q$ is a 1–1 correspondence between the stable models of $\Gamma \cup \{Def(q) \rightarrow q : q \in Q\}$ and the stable models of Γ .*

Proof. Let Γ_2 be $\{Def(q) \rightarrow q : q \in Q\}$. Since Q contains all head atoms of Γ_2 but no atom occurring in Γ then, by the splitting set theorem (Proposition 10), (“s.m.” stands for “a stable model”)

$$X \text{ is s.m. of } \Gamma \cup \Gamma_2 \text{ iff } X \setminus Q \text{ is s.m. of } \Gamma \text{ and } X \text{ is s.m. of } (X \setminus Q) \cup \Gamma_2. \quad (36)$$

Clearly, if X is a stable model of $\Gamma \cup \Gamma_2$ then $X \setminus Q$ is a stable model of Γ , which proves one of the two directions of the 1–1 correspondence in the claim. Now take any stable model Y of Γ . We need to show that there is exactly one stable model X of $\Gamma \cup \Gamma_2$ such that $X \setminus Q = Y$. In view of (36), it is sufficient to show that

$$Z = Y \cup \{q \in Q : Y \models Def(q)\}$$

is the only stable model X of $Y \cup \Gamma_2$, and that $Z \setminus Q = Y$. This second condition can be easily verified. Now consider $Y \cup \Gamma_2$. By Lemma 8, $Y \cup \Gamma_2$ has the same stable models of

$$Y \cup \{Def(q)' \rightarrow q : q \in Q\},$$

where $Def(q)'$ is obtained from $Def(q)$ by replacing all occurrences of atoms in it with \top if the atom replaced belongs to Y , and with \perp otherwise. This theory can be further simplified into theory Z . Indeed, $Def(q)'$ doesn't contain atoms, and then it is strongly equivalent to \top or \perp . In particular, if $Y \models Def(q)$ then $Def(q)'$ is strongly equivalent to \top , and then $Def(q)' \rightarrow q$ is strongly equivalent to q . Otherwise, $Def(q)'$ is strongly equivalent to \perp , and then $Def(q)' \rightarrow q$ is strongly equivalent to \top . As Z is a set of atoms, it is easy to verify that its only stable model is Z itself. \square

4.5 Proof of Proposition 9

In order to prove the Completion Lemma, we will need the following lemma.

Lemma 9. *Take any two sets X, Y of atoms such that $Y \subseteq X$. For any formula F and any set S of atoms,*

- (a) *if each positive occurrence of an atom from S in F is in the scope of negation and $Y \models F^X$ then $Y \setminus S \models F^X$, and*
- (b) *if each negative occurrence of an atom from S in F is in the scope of negation and $Y \setminus S \models F^X$ then $Y \models F^X$.*

- Proof.*
- If $X \not\models F$ then $F^X = \perp$, and the claim is trivial. This covers the case in which $F = \perp$.
 - If $X \models F$ and F is an atom a then claim (b) holds because if $a \in Y \setminus S$ then $a \in Y$. For claim (a), if $a \notin S$ and $a \in Y$ then $a \in Y \setminus S$.
 - If $X \models F$ and F is a conjunction or a disjunction, the claim is almost immediate by Lemma 2 and induction hypothesis.
 - The case in which $X \models F$ and F has the form $G \rightarrow H$ remains. Clearly, $(G \rightarrow H)^X = G^X \rightarrow H^X$. **Case 1.** If $G \rightarrow H$ is a negation (that is, $H = \perp$) then, since $X \models F$, $X \not\models G^X$ and then $F^X = \top$, and the claims clearly follows. **Case 2:** $H \neq \perp$. We describe a proof of claim (a). The proof for (b) is similar. Assume that no atom from S has positive occurrences in $G \rightarrow H$ outside the scope of the negation, that $Y \models G^X \rightarrow H^X$, and that $Y \setminus S \models G^X$. We want to prove that $Y \setminus S \models H^X$. Notice that no atom from S has negative occurrences in G outside the scope of negation; consequently, by the induction hypothesis (claim (b)), $Y \models G^X$. On the other hand, $Y \models (G \rightarrow H)^X$, so that $Y \models H^X$. Since no atom from S has positive occurrences in H outside the scope of negation, we can conclude that $Y \setminus S \models H^X$ by induction hypothesis (claim (a)).

□

Proposition 9 (Completion Lemma) *Let Γ be any propositional theory, and Q a set of atoms that have positive occurrences in Γ only in the scope of negation. For each $q \in Q$, let $Def(q)$ be a formula such that all negative occurrences of atoms from Q in $Def(q)$ are in the scope of negation. Then $\Gamma \cup \{Def(q) \rightarrow q : q \in Q\}$ and $\Gamma \cup \{Def(q) \leftrightarrow q : q \in Q\}$ have the same stable models.*

Proof. Let Γ_1 be $\Gamma \cup \{Def(q) \rightarrow q : q \in Q\}$ and let Γ_2 be $\Gamma \cup \{q \rightarrow Def(q) : q \in Q\}$. We want to prove that a set X of atoms is a stable model of both theories or for none of them. Since $\Gamma_1^X \subseteq \Gamma_2^X$, Γ_2^X entails Γ_1^X . If the opposite entailment holds also then we clearly have that Γ_2^X and Γ_1^X are satisfied by the same subsets of X , and the claim immediately follows. Otherwise, for some $Y \subseteq X$, $Y \not\models \Gamma_2^X$ and $Y \models \Gamma_1^X$. First of all, that means that $X \models \Gamma_1$, so that Γ_1^X is equivalent to

$$\Gamma^X \cup \{Def(q)^X \rightarrow q : q \in Q \cap X\}.$$

Secondly, set Y' is one of the sets Y' having the following properties:

- (i) $Y' \setminus Q = Y \setminus Q$, and
- (ii) $Y' \models Def(q)^X \rightarrow q$ for all $q \in Q \cap X$.

Let Z be the intersection of such sets Y' , and let Δ be $\{q \rightarrow Def(q)^X : q \in Q \cap X\}$. Set Z has the following properties:

- (a) $Z \subseteq Y$,
- (b) $Z \models \Gamma_1^X$, and
- (c) $Z \models \Delta$.

Indeed, claim (a) holds since Y is one of the elements Y' of the intersection. To prove (b), first of all, we observe that $Z \setminus Q = Y \setminus Q$, so that, by (a), there is a set $S \subseteq Q$ such that $Z = Y \setminus S$; as $Y \models \Gamma^X$ and Γ has all positive occurrences of atoms from $S \subseteq Q$ in the scope of negation, it follows that $Z \models \Gamma^X$ by Lemma 9(a). It remains to show that, for any q , if $Z \models Def(q)^X$ then $q \in Z$. Assume that $Z \models Def(q)^X$. Then, since $Def(q)$ has all negative occurrences of atoms from Q in the scope of negation, and since all Y' whose intersection generate Z are superset of Z with $Y' \setminus Z \subseteq Q$, all those Y' satisfy $Def(q)^X$ by Lemma 9. By property (ii), we have that $q \in Y'$ for all Y' , and then $q \in Z$.

It remains to prove claim (c). Take any $q \in Z$ that belongs to $Q \cap X$. Set $Y' = Z \setminus \{q\}$ satisfies condition (i), but it cannot satisfy (ii), because sets Y' that satisfy (i) and (ii) are supersets of Z by construction of Z . Consequently, $Y' \not\models Def(q)^X$. Since all positive occurrences of atom q in $Def(q)$ are in the scope of negation and $Y' = Z \setminus \{q\}$, we can conclude that $Z \not\models Def(q)^X$ by Lemma 9 again.

Now consider two cases. If $X \not\models \Gamma_2$ then clearly X is not a stable model of Γ_2 . It is not a stable model of Γ_1 as well. Indeed, since $X \models \Gamma_1$, we have that, for some $q \in Q \cap X$, $X \not\models Def(q)$. Consequently, $Def(q)^X = \perp$ and then $X \not\models \Delta$, but, since $Z \models \Delta$ by (c) and $Z \subseteq Y \subseteq X$ by (a), Z is a proper subset of X . Since $Z \models \Gamma_1^X$ by (b), X is not a stable model of Γ_1 .

In the other case ($X \models \Gamma_2$) it is not hard to see that Γ_2^X is equivalent to $\Gamma_1^X \cup \Delta$. We have that $Z \models \Gamma_1^X$ by (b), and then $Z \models \Gamma_2^X$ by (c). Since $Y \not\models \Gamma_2^X$, $Z \neq Y$. On the other hand, $Z \subseteq Y \subseteq X$ by (a). This means that Z is a proper subset of X that satisfies Γ_1^X and Γ_2^X , and we can conclude that X is not an stable model of any of Γ_1 and Γ_2 . \square

4.6 Proof of Proposition 11

Lemma 1. Rule

$$l_1 \wedge \cdots \wedge l_m \rightarrow a_1 \vee \cdots \vee a_n \quad (37)$$

($n > 0, m \geq 0$) where a_1, \dots, a_n are atoms and l_1, \dots, l_m are literals, is strongly equivalent to the set of n implications ($i = 1, \dots, n$)

$$(l_1 \wedge \cdots \wedge l_m \wedge (a_1 \rightarrow a_i) \wedge \cdots \wedge (a_n \rightarrow a_i)) \rightarrow a_i. \quad (38)$$

Proof. Let F be (37) and G_i ($i = 1, \dots, n$) be (38). We want to prove that F is strongly equivalent to $\{G_1, \dots, G_n\}$ by showing that F^X is classically equivalent to $\{G_1^X, \dots, G_n^X\}$. Let H be $l_1 \wedge \dots \wedge l_m$.

Case 1: $X \not\models H$. Then the antecedents of F and of all G_i are not satisfied by X . It is then easy to verify that the reducts of F and of all G_i relative to X are equivalent to \top . **Case 2:** $X \models H$ and $X \not\models F$. Then clearly $F^X = \perp$. But, for each i , G_i^X is \perp : indeed, since $X \not\models F$, $X \not\models a_i$ for all $i = 1, \dots, n$. It follows that the consequent of each G_i is not satisfied by X , but the antecedent is satisfied, because $X \models H$ and in each implication $a_j \rightarrow a_i$ in G_i , the antecedent is not satisfied. **Case 3:** $X \models H$ and $X \models F$. This means that some of a_1, \dots, a_n belong to X . Assume, for instance, that a_1, \dots, a_p ($0 < p \leq n$) belong to X , and a_{p+1}, \dots, a_n don't. Then F^X is equivalent to $H^X \rightarrow (a_1 \vee \dots \vee a_p)$. Now consider formula G_i . If $i > p$ then the consequent a_i is not satisfied by X , but also the antecedent is not: it contains an implication $a_1 \rightarrow a_i$; consequently G_i^X is \top . On the other hand, if $i \leq p$ then the consequent a_i is satisfied by X , as well as each implication $a_j \rightarrow a_i$ in the antecedent of G_i . After a few simplifications, we can rewrite G_i^X as

$$(H^X \wedge (a_1 \rightarrow a_i) \wedge \dots \wedge (a_p \rightarrow a_i)) \rightarrow a_i.$$

It is not hard to see that this formula is classically equivalent to

$$(H^X \rightarrow (a_1 \vee \dots \vee a_p))$$

which is equivalent to F^X , so that the claim easily follows. \square

Proposition 11. *The problem of the existence of a stable model of a theory consisting of formulas of the form $F \rightarrow a$ and $F \rightarrow \perp$ is Σ_2^P -hard.*

Proof. The problem is in class Σ_2^P because, as mentioned in Section sec:prop-compl, the same problem for the (larger) class of arbitrary theories is also in Σ_2^P [Pearce *et al.*, 2001]. Hardness remains to be proven.

In view of Lemma 1, we can transform a disjunctive program into a theory consisting of formulas of the form $F \leftarrow a$, with the same stable models and in polynomial time. Consequently, as the existence of a stable model of a disjunctive program is Σ_2^P -hard by [Eiter and Gottlob, 1993], the same holds for theories as in the statement of this proposition. \square

4.7 Proof of Propositions 12 and 13

For the proof of these propositions, we define an *extended aggregate* to be either an aggregate of the form (6), or \perp . It is easy to see, that, for each aggregate A of the form (6) and any set X of atoms, A^X is an extended aggregate. We also define, for any extended aggregate A , \hat{A} as

- the formula (9) if A has the form (6), and
- \perp , otherwise.

Lemma 10. *For any extended aggregate A , \hat{A} is classically equivalent to A .*

Proof. The case $A = \perp$ is trivial. The remaining case is when A is an aggregate. Consider any possible conjunctive term H_I (where $I \subseteq \{1, \dots, n\}$) of \hat{A} :

$$\left(\bigwedge_{i \in I} F_i \right) \rightarrow \left(\bigvee_{i \in \bar{I}} F_i \right).$$

For each set X of atoms there is exactly one set I such that $X \not\models H_I$: the set I_X that consists of the i 's such that $X \models F_i$. Consequently, for every set X of atoms,

$$\begin{aligned} X \models \hat{A} & \text{ iff } H_{I_X} \text{ is not a conjunctive term of } \hat{A} \\ & \text{ iff } op(\{w_i : i \in I_X\}) \prec N \\ & \text{ iff } op(\{w_i : X \models F_i\}) \prec N \\ & \text{ iff } X \models A. \end{aligned}$$

□

Lemma 11. *For any aggregate A and any set X of atoms, \hat{A}^X is classically equivalent to \hat{A}^X .*

Proof. **Case 1:** $X \not\models A$. Then $\hat{A}^X = \hat{\perp} = \perp$. On the other hand, by Lemma 10, $X \not\models \hat{A}$ so that $\hat{A}^X = \perp$ also. **Case 2:** $X \models A$. Then A is an aggregate, and, by the definition of a reduct, \hat{A}^X is

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\left(\bigwedge_{i \in I} F_i^X \right) \rightarrow \left(\bigvee_{i \in \bar{I}} F_i^X \right) \right). \quad (39)$$

On the other hand, \hat{A}^X is classically equivalent, by Lemma 2, to

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\left(\bigwedge_{i \in I} F_i \right) \rightarrow \left(\bigvee_{i \in \bar{I}} F_i \right) \right)^X.$$

Notice that, since $X \models \hat{A}$ by Lemma 10, all implications in the formula above are satisfied by X . Consequently, \hat{A}^X is classically equivalent to

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\left(\bigwedge_{i \in I} F_i \right)^X \rightarrow \left(\bigvee_{i \in \bar{I}} F_i \right)^X \right),$$

and then, by Lemma 2 again, to (39). \square

Proposition 12. *Let A be an aggregate of the form (6) and let G be the corresponding formula (9). Then*

- (a) G is classically equivalent to A , and
- (b) for any set X of atoms, G^X is classically equivalent to A^X .

Proof. Part (a) is immediate from Lemma 10, as $G = \hat{A}$. For part (b), we need to show that \hat{A}^X is classical equivalent to A^X . By Lemma 11, \hat{A}^X is classically equivalent to \hat{A}^X . It remains to notice that \hat{A}^X is classically equivalent to A^X by Lemma 10. \square

Lemma 12. *For any aggregate $op(\{F_1 = w_1, \dots, F_n = w_n\}) \prec N$, formula (9) is classically equivalent to*

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\bigvee_{i \in \bar{I}} F_i \right) \quad (40)$$

if the aggregate is monotone, and to

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\neg \bigwedge_{i \in I} F_i \right)$$

if the aggregate is antimonotone.

Proof. Consider the case of a monotone aggregate first. Let G be (9), and H be (40). It is easy to verify that H entails G . The opposite direction remains. Assume G , and we want to derive every conjunctive term

$$\bigvee_{i \in \bar{I}} F_i \quad (41)$$

in H . For every conjunctive term D of the form (41) in H , $op(\{w_i : i \in I\}) \not\prec N$. As the aggregate is monotone then, for every subset I' of I , $op(\{w_i : i \in I'\}) \not\prec N$, so that the implication

$$\left(\bigwedge_{i \in I'} F_i \right) \rightarrow \left(\bigvee_{i \in \bar{I}} F_i \right)$$

is a conjunctive term of H for all $I' \subseteq I$. Then, since $\bar{I}' = \bar{I} \cup (I \setminus I')$, (“ \Rightarrow ” denotes entailment, and “ \Leftrightarrow ” equivalence)

$$\begin{aligned} H &\Rightarrow \bigwedge_{I' \subseteq I} \left(\left(\bigwedge_{i \in I'} F_i \right) \rightarrow \left(\bigvee_{i \in \bar{I}'} F_i \right) \right) \\ &\Leftrightarrow \bigwedge_{I' \subseteq I} \left(\left(\left(\bigwedge_{i \in I'} F_i \right) \wedge \bigwedge_{i \in I' \setminus I} \neg F_i \right) \rightarrow \left(\bigvee_{i \in \bar{I}'} F_i \right) \right) \\ &\Leftrightarrow \left(\bigvee_{I' \subseteq I} \left(\left(\bigwedge_{i \in I'} F_i \right) \wedge \bigwedge_{i \in I' \setminus I} \neg F_i \right) \right) \rightarrow D. \end{aligned}$$

The antecedent of the implication is a tautology: for each interpretation X , the disjunctive term relative to $I' = \{i \in I : X \models F_i\}$ is satisfied by X . We can conclude that H entails D .

The proof for antimonotone aggregates is similar. \square

Proposition 13. *For any aggregate $op\langle\{F_1 = w_1, \dots, F_n = w_n\}\rangle \prec N$, formula (9) is strongly equivalent to*

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\bigvee_{i \in \bar{I}} F_i \right)$$

if the aggregate is monotone, and to

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\neg \bigwedge_{i \in I} F_i \right)$$

if the aggregate is antimonotone.

Proof. Consider the case of a monotone aggregate first. Let G be (9), and H be (40). In view of Proposition 5, it is sufficient to show that G^X is equivalent to H^X in classical logic for all sets X . If $X \not\models H$ then also $X \not\models G$ by Lemma 12, so that both reducts are \perp . Otherwise ($X \models H$), by the same lemma, $X \models G$. Then, by Lemma 11, G^X is classically equivalent to (39). On the other hand, it is easy to verify, by applying Lemma 2 to H^X twice, that H^X is classically equivalent to

$$\bigwedge_{I \subseteq \{1, \dots, n\} : op(\{w_i : i \in I\}) \not\prec N} \left(\bigvee_{i \in \bar{I}} F_i^X \right).$$

The claim now follows from Lemma 12.

The reasoning for nonmonotone aggregates is similar. \square

4.8 Proof of Proposition 14

Let Γ be the theory consisting of formulas (17)–(20).

Lemma 13. *For any stable model X of Γ , X contains an atom s_i iff X contains an atom b_j such that bid j involves selling object i .*

Proof. Consider Γ as a propositional theory. We notice that

- formulas (19) can be strongly equivalently grouped as m formulas ($i = 1, \dots, m$)

$$\left(\bigwedge_{j=1, \dots, n: \text{object } i \text{ is part of bid } j} b_j \right) \rightarrow s_i,$$

and

- no other formula of Γ contains atoms of the form s_i outside the scope of negation.

Consequently, by the Completion Lemma (Proposition 9), formulas (19) in Γ can be replaced by m formulas ($i = 1, \dots, m$)

$$\left(\bigwedge_{j=1, \dots, n: \text{object } i \text{ is part of bid } j} b_j \right) \leftrightarrow s_i. \quad (42)$$

preserving the stable models. It follows that every stable model of Γ must satisfy formulas (42), and the claim immediately follows. \square

Proposition 14. *$X \mapsto \{i : b_i \in X\}$ is a 1–1 correspondence between the stable models of the theory consisting of formulas (17)–(20) and a solution to Joe’s problem.*

Proof. Take any stable model X of Γ . Since X satisfies rules (18) of Γ , condition (a) is satisfied. Condition (b) is satisfied as well, because X contains exactly all atoms s_i sold in some bids by Lemma 13, and since X satisfies aggregate (20) that belongs to Γ .

Now consider a solution of Joe’s problem. This determines which atoms of the form b_i belongs to a possible corresponding stable model X . Consequently, Lemma 13 determines also which atoms of the form s_j belong to X , reducing the candidate stable models X to one. We need to show that this X is indeed a stable model of Γ . The reduct Γ^X consists of (after a few simplifications)

- (i) all atoms b_i that belong to X (from (17)),
- (ii) \top from (18) since (a) holds,

- (iii) (by Lemma 13) implications (19) such that both b_j and s_i belong to X , and
- (iv) the reduct of (20) relative to X .

Notice that (i)–(iii) together are equivalent to X , so that every every proper subset of X doesn't satisfy Γ^X . It remains to show that $X \models \Gamma^X$. Clearly, X satisfies (i)–(iii). To show that X satisfies (iv) it is sufficient, by Lemma 5 (consider (20) as a propositional formula), to show that X satisfies (20): it does that by hypothesis (b). \square

4.9 Proof of Propositions 15 and 16

Lemma 14. *If, for every aggregate, computing $op(W) \prec N$ requires polynomial time then*

- (a) *checking satisfaction of a theory with aggregates requires polynomial time, and*
- (b) *computing the reduct of a theory with aggregates requires polynomial time.*

Proof. Part (a) is easy to verify by structural induction. Computing the reduct essentially consists of checking satisfaction of subexpressions of each formula of the theory. Each check doesn't require too much time by (a). It remains to notice that each formula with aggregates has a linear number of subformulas. \square

Proposition 15. *If, for every aggregate, computing $op(W) \prec N$ requires polynomial time then the existence of a stable model of a theory with aggregates is a Σ_2^P -complete problem.*

Proof. Hardness follows from the fact that theories with aggregates are a generalization of theories without aggregates. To prove inclusion, consider that the existence of a stable model of a theory Γ is equivalent to satisfiability of:

exists X such that **for all** Y , if $Y \subseteq X$ then $Y \models \Gamma^X$ iff $X = Y$

It remains to notice that, in view of Lemma 14, checking (for any X and Y)

if $Y \subseteq X$ then $Y \models \Gamma^X$ iff $X = Y$

requires polynomial time. \square

Lemma 15. *Let F be a formula with aggregates containing monotone and anti-monotone aggregates only, no equivalences and no implications other than negations. For any sets X, Y and Z such that $Y \subseteq Z$, if $Y \models F^X$ then $Z \models F^X$.*

```

function verifyAS( $\Gamma, X$ )
  if  $X \not\models \Gamma$  then return false
   $\Delta := \{F^X \rightarrow a : F \rightarrow a \in \Gamma \text{ and } X \models a\}$ 
   $Y := \emptyset$ 
  while there is a formula  $G \rightarrow a \in \Delta$  such that  $Y \models G$  and  $a \notin Y$ 
     $Y := Y \cup \{a\}$ 
  end while
  if  $Y = X$  then return true
  return false

```

Figure 5: A polynomial-time algorithm that checks stable models of special kinds of theories

Proof. Let G be F with each monotone aggregate replaced by (15) and each antimonotone aggregate replaced by (16). It is easy to verify that G is a nested expression. Nested expressions have all negative occurrences of atoms in the scope of negation, so if $Y \models G^X$ then $Z \models G^X$ by Lemma (9). It remains to notice that F^X and G^X are satisfied by the same sets of atoms by Propositions 13 and 12. \square

Proposition 16. *Consider theories with aggregates consisting of formulas of the form*

$$F \rightarrow a, \quad (43)$$

where a is an atom or \perp , and F contains monotone and antimonotone aggregates only, no equivalences and no implications other than negations. If, for every aggregate, computing $op(W) \prec N$ requires polynomial time then the problem of the existence of a stable model of theories of this kind is an NP-complete problem.

Proof. NP-hardness follows from the fact that theories with aggregates are a generalization of traditional programs, for which the same problem is NP-complete. For inclusion in NP, it is sufficient to show that the time required to check if a set X of atoms is a stable model of Γ is polynomial. An algorithm that does this test is in Figure 5. It is easy to verify that it is a polynomial time algorithm. It remains to prove that it is correct. If $X \not\models \Gamma$ then it is trivial. Now assume that $X \models \Gamma$. It is sufficient to show that

- (a) Δ is classically equivalent to Γ^X , and
- (b) the last value of Y (we call it Z) is the unique minimal model of Δ .

Indeed, for part (a), we notice that, since $X \models \Gamma$, Γ^X is

$$\{F^X \rightarrow a^X : F \rightarrow a \in \Gamma \text{ and } X \models a\} \cup \{F^X \rightarrow a^X : F \rightarrow a \in \Gamma \text{ and } X \not\models a\}.$$

The first set is Δ . The second set (which includes the case in which $a = \perp$) is a set of $\perp \rightarrow \perp$. Indeed, each $a^X = \perp$, and since $X \models \Gamma$, X doesn't satisfy any F and then $F^X = \perp$.

For part (b) it is easy to verify that the **while** loop iterates as long as $Y \not\models \Delta$, so that $Z \models \Delta$. Now assume, in sake of contradiction, that there is a set Z' that satisfies Δ and that is not a superset of Z . Consider, in the execution of the algorithm, the first atom $a \notin Z'$ added to Y , and that value of $Y \subseteq Z'$ to which a has been added to. This means that Δ contains a formula $G \rightarrow a$ such that $Y \models G$. Recall that G stands for a formula of the form F^X , where F is a formula with aggregates with monotone and antimonotone aggregates only and without implications (other than negations) or equivalences. Consequently, by Lemma 15, $Z' \models G$. On the other hand, $a \notin Z'$, so $Z' \not\models G \rightarrow a$, contradicting the hypothesis that Z' is a model of Δ . \square

4.10 Proof of Proposition 17

Lemma 16. *Let F and G two propositional formulas, and let F' and G' the result of replacing each occurrence of an atom a in F and G with a propositional formula H . If F and G are strongly equivalent to each other then F' and G' are strongly equivalent to each other.*

Proof. It follows from Proposition 5, in view of the following fact: if F and G are equivalent in the logic of here-and-there to each other then F' and G' are equivalent in the logic of here-and-there to each other. \square

Lemma 17. *Let F and G be two propositional formulas that are AND-OR combinations of \top , \perp and atoms only. If F and G are classically equivalent to each other then they are strongly equivalent to each other also.*

Proof. In view of Proposition 5, it is sufficient to show that, for every set X of atoms, F^X is classically equivalent to G^X . By Lemma 2 we can distribute the reduct operator in F^X to its atoms. It follows that F^X is classically equivalent to F with all occurrences of atoms that don't belong to X replaced by \perp , and similarly for G^X . The fact that F^X is classically equivalent to G^X now follows from the classical equivalence between F and G . \square

Next Lemma immediately follows from our definition of satisfaction of aggregates (Section 3.1 of this paper), and the definition of $[L \leq S]$ and $[S \leq U]$ and Proposition 1 from [Ferraris and Lifschitz, 2005b].

Lemma 18. *For every weight constraints $L \leq S$ and $S \leq U$ and any set X of atoms,*

(a) $X \models [L \leq S]$ iff $X \models \text{sum}\langle S \rangle \geq L$, and

(b) $X \models [S \leq U]$ iff $X \models \text{sum}\langle S \rangle \leq U$.

Proposition 17. *In presence of nonnegative weights only, $[N \leq S]$ is strongly equivalent to $\text{sum}\langle S \rangle \geq N$, and $[S \leq N]$ is strongly equivalent to $\text{sum}\langle S \rangle \leq N$.*

Proof. We start with (a), with the special case when rule elements F_1, \dots, F_n of S are distinct atoms. Since the aggregate is monotone then, by Lemma 13, we just need to show that $[N \leq S]$ is strongly equivalent to (15). As classical equivalence holds between $[N \leq S]$ and $\text{sum}\langle S \rangle \geq N$ by Lemma 18, the same relationship holds between $[N \leq S]$ and (15). As both formulas are AND-OR combinations of atoms, the claim follows by Lemma 17. The most general case of (a) follows from the special case, by Lemma 16.

For part (b), we know, by Lemma 13, that antimonotone aggregate $\text{sum}\langle S \rangle \leq U$ (written as a formula (6)) is strongly equivalent to formula

$$\bigwedge_{I \subseteq \{1, \dots, n\} : \sum_{i \in I} w_i > U} (\neg \bigwedge_{i \in I} F_i).$$

By applying DeMorgan's law to this last formula (which preserves equivalence in the logic of here-and-there and then it is a strongly equivalent transformation by Proposition 5) we get $S \leq U$. \square

4.11 Proof of Proposition 18

Given a PDB-aggregate of the form (6) and a set X of literals, by I_X we denote the set $\{i \in \{1, \dots, n\} : X \models F_i\}$.

Lemma 19. *For each PDB-aggregate of the form (6), a set X of atoms satisfies a formula of the form $G_{(I_1, I_2)}$ iff $I_1 \subseteq I_X \subseteq I_2$.*

Proof.

$$\begin{aligned} X \models G_{(I_1, I_2)} & \text{ iff } X \models F_i \text{ for all } i \in I_1, \text{ and } X \not\models F_i \text{ for all } i \in \{1, \dots, n\} \setminus I_2 \\ & \text{ iff } X \models F_i \text{ for all } i \in I_1, \text{ and for every } i \text{ such that } X \models F_i, i \in I_2 \\ & \text{ iff } I_1 \subseteq I_X \text{ and } I_X \subseteq I_2. \end{aligned}$$

\square

Lemma 20. *For every PDB-aggregate A , A_{tr} is classically equivalent to (9).*

Proof. Consider a set X of atoms. By Lemma 19, $X \models A_{tr}$ iff

X satisfies one of the disjunctive terms $G_{(I_1, I_2)}$ of A_{tr}

and then iff

A_{tr} contains a disjunctive term $G_{(I_1, I_2)}$ such that $I_1 \subseteq I_X \subseteq I_2$.

It is easy to verify that if this condition holds then one of such terms $G_{(I_1, I_2)}$ is $G_{(I_X, I_X)}$. Consequently,

$$\begin{aligned} X \models A_{tr} & \text{ iff } A_{tr} \text{ contains disjunctive term } G_{(I_X, I_X)} \\ & \text{ iff } op(W_{I_X}) \prec N. \end{aligned}$$

We have essentially found that $X \models A_{tr}$ iff $X \models A$. The claim now follows by Proposition 12(a). \square

Lemma 21. *For any PDB-aggregate A , A_{tr} is strongly equivalent to*

(a)

$$\bigvee_{I \in \{1, \dots, n\}: op(W_I) \prec N} G_{(I, \{1, \dots, n\})}$$

if A is monotone, and to

(b)

$$\bigvee_{I \in \{1, \dots, n\}: op(W_I) \prec N} G_{(\emptyset, I)}$$

if it is antimonotone.

Proof. To prove (a), assume that A is monotone. Then, if A_{tr} contains a disjunctive term $G_{(I_1, I_2)}$ then it contains the disjunctive term $G_{(I_1, \{1, \dots, n\})}$ as well. Consider also that formula $G_{(I_1, \{1, \dots, n\})}$ entails $G_{(I_1, I_2)}$ in the logic of here-and-there. Then, by Proposition 5, we can drop all disjunctive terms of the form $G_{(I_1, I_2)}$ with $I_2 \neq \{1, \dots, n\}$, preserving strong equivalence. Formula A_{tr} becomes

$$\bigvee_{I_1 \subseteq \{1, \dots, n\}: \text{for all } I \text{ such that } I_1 \subseteq I \subseteq \{1, \dots, n\}, op(W_I) \prec N} G_{(I_1, \{1, \dots, n\})}.$$

It remains to notice that, since A is monotone, if $op(W_{I_1}) \prec N$ then $op(W_I) \prec N$ for all I superset of I_1 .

The proof for (b) is similar. \square

Proposition 18 *For any monotone or antimonotone PDB-aggregates A of the form (6) where F_1, \dots, F_n are atoms, A_{tr} is strongly equivalent to (9).*

Proof. Let S be $\{F_1 = w_1, \dots, F_n = w_n\}$. Lemma 20 says that A_{tr} is classically equivalent to (9) for every formulas F_1, \dots, F_n in S . We can then prove the claim of this proposition using Lemma 17, by showing that both A_{tr} and (9) can be strongly equivalently rewritten as AND-OR combinations of

- $F_1, \dots, F_n, \top, \perp$, if A is monotone, and
- $\neg F_1, \dots, \neg F_n, \top, \perp$, if A is antimonotone.

About (9), this has already been shown in the proof of Proposition 17, while, about A_{tr} , this is shown by Lemma 21. Indeed, each $G_{(I, \{1, \dots, n\})}$ is a (possibly empty) conjunction of terms of the form F_i , and each $G_{(\emptyset, I)}$ is a (possibly empty) conjunction of terms of the form $\neg F_i$, since each F_i is an atom. \square

4.12 Proof of Proposition 19

We observe, first of all, that the definition of satisfaction of FLP-aggregates and FLP-programs in [Faber *et al.*, 2004] is equivalent to ours. The definition of a reduct is different, however. Next lemma is easily provable by structural induction.

Lemma 22. *For any nested expression F without negations and any two sets X and Y of atoms such that $Y \subseteq X$, $Y \models F^X$ iff $Y \models F$.*

Lemma 23. *For any FLP-aggregate A and any set X of atoms, if $X \models A$ then*

$$Y \models A^X \text{ iff } Y \models A.$$

Proof. Let A have the form (6). Since $X \models A$, A^X has the form

$$op(\{F_1^X = w_1, \dots, F_n^X = w_n\}) \prec N.$$

In case of FLP-aggregates, each F_i is a conjunction of atoms. Then, by Lemma 22, $Y \models F_i^X$ iff $Y \models F_i$. The claim immediately follows from the definition of satisfaction of aggregates. \square

Proposition 19. *The stable models of a positive FLP-program under our semantics are identical to its stable models in the sense of [Faber *et al.*, 2004].*

Proof. It is easy to see that if $X \not\models \Pi$ then $X \not\models \Pi^X$ and $X \not\models \Pi^{\underline{X}}$, so that X is not a stable model under either semantics. Now assume that $X \models \Pi$. We will show that the two reducts are satisfied by the same subsets of X . It is sufficient to consider the case in which Π contains only one rule

$$A_1 \wedge \cdots \wedge A_m \rightarrow a_1 \vee \cdots \vee a_n. \quad (44)$$

If $X \not\models A_1 \wedge \cdots \wedge A_m$ then $\Pi^{\underline{X}} = \emptyset$, and Π^X is the tautology

$$\perp \rightarrow (a_1 \vee \cdots \vee a_n)^X.$$

Otherwise, $\Pi^{\underline{X}}$ is rule (44), and Π^X is

$$A_1^X \wedge \cdots \wedge A_m^X \rightarrow (a_1 \vee \cdots \vee a_n)^X.$$

These two reducts are satisfied by the same subsets of X by Lemmas 22 and 23. \square

5 Conclusions

We have proposed a new definition of stable model — for proposition theories — that is simple, very general, and that inherits several properties from logic programs with nested expressions. On top of that, we have defined the concept of an aggregate, both as an atomic operator and as a propositional formula. We hope that this very general framework may be useful in the heterogeneous world of aggregates in answer set programming.

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