CS 380C: Problem set on Fixpoint Equations
Due date: February 7th, 2013

January 30, 2013

In this assignment, the word *domain* refers to a finite set S with a partial order $\leq (\subseteq S \times S)$ under which there is a least element. We will write $D = (S, \leq)$ to represent the domain.

1. Consider the domain D that is the powerset of the set {a,b,c}, in which elements are ordered by subset ordering, and {} is the least element.

   (a) Draw a diagram of this partially ordered set. You do not have to show transitive edges.

   (b) Write down a function D→ D that is monotonic but not extensive.

   (c) Write down a function D→ D that is extensive but not monotonic.

   (d) Write down a function D→ D that is both extensive and monotonic.

   (e) Write down a function $f : D \to D$ for which the equation $x = f(x)$ has no solutions. Explain why your function is not monotonic.

   (f) Write down a function $f : D \to D$ for which the equation $x = f(x)$ has multiple solutions.

2. Let $D = (S, \leq)$ be a domain, and let $f : D \to D$ be monotonic. Let set $L \subseteq S$ be the set of solutions to the equation $x = f(x)$. Show that $(L, \leq)$ is itself a domain.

3. Let $D = (\mathscr{S}, \subseteq)$ be the domain consisting of the powerset of a set S ordered by subset ordering, and let $a, b \in S$. Consider the function $f : \mathscr{S} \to \mathscr{S}$ defined as follows: $f(T) =$ if $(a \in T)$ then $(T \cup \{b\})$ else T. Argue that f is monotonic.

   How about the function $g(T) =$ if $(a \in T)$ then $(T - \{a\})$ else T ?

4. If D is a domain and f:D→D and g:D→D are monotonic, show that the function h(x) = f(g(x)) is monotonic.

5. Let D be a domain and let f:D→D and g:D→D be monotonic and extensive functions.

(a) Show that any solution to the following system of simultaneous equations:

x = f(x)
x = g(x)

is a solution to the equation x = f(g(x)) and vice versa.

(b) From this and the result of (2), argue that this system of simultaneous equations always has a least solution, and describe how to compute it.

(c) Do the results of (a) and (b) always hold if f and g are monotonic but not extensive?

(d) Do the results of (a) and (b) always hold if f and g are extensive but not monotonic?

6. A non-terminal in a context-free grammar is said to be *nullable* if the empty string ϵ can be derived from that non-terminal by repeatedly applying the rules of the grammar. For example, in the following grammar, A, B and C are nullable, while S is not.

$$\begin{array}{lcl} S & \rightarrow & aA \\ A & \rightarrow & BC \\ A & \rightarrow & x \\ B & \rightarrow & \epsilon \\ C & \rightarrow & \epsilon \end{array}$$

(a) For a given context-free grammar, let NULL be the set of nullable non-terminals. Show that NULL can be expressed as the least solution of a system of simultaneous equational constraints over the powerset of the non-terminals of the grammar, in which there is one equation for each rule of the grammar. (Hint: a rule A $\rightarrow \epsilon$ would generate the equational constraint NULL = NULL $\cup$ {A}).

(b) Using the result of problem (3), describe informally how such a system of equational constraints might be solved. Use your algorithm to find the NULL set for the sample grammar.

7. The theory of LL(1) grammars is based on two relations called FIRST and FOLLOW. These relations are usually described in a very complicated way in text-books, but they can be computed very easily if you use the theory of fixpoint equations. You do not need to know anything about LL(1) grammars to do this problem.

(a) If A is a non-terminal in a context-free grammar G, the set FIRST(A) is the set of terminal symbols t such that A $\stackrel{+}{\rightarrow}$ t α (that is, t is the first terminal symbol in a string that can be derived from A). If A is nullable, ϵ is in FIRST(A) by convention. Show that the FIRST sets of the non-terminals of a grammar can be expressed as the solutions

of a system of equational constraints derived from the grammar rules. You can assume that the set of nullable non-terminals is given to you.

(b) If A is a non-terminal in a context-free grammar, the set FOLLOW(A) is the set of terminals t such as that S $\stackrel{+}{\rightarrow}$ α A t β (that is, t follows A in a string that can be derived from the start symbol S). Show that the FOLLOW sets of the non-terminals of a grammar can be expressed as the solutions of a system of equational constraints derived from the grammar rules. For this part, you can assume that the FIRST sets are given to you.

8. In the fixpoint theorem we proved in class for domains D consisting of finite partially-ordered sets with a least element $\perp$, we considered monotonic functions f and we considered an iteration of the form $\perp, f(\perp), f(f(\perp)), ...$

   If a is an *arbitrary* element of the domain D, and f is an arbitrary function (not necessarily monotonic), what is the strongest statement you can make about the sequence $a, f(a), f(f(a)), ...$? What can you say if f is monotonic?

9. In this problem, you need to formulate dataflow equations to solve certain problems. You need to specify (i) the domain, (ii) whether the problem is forward or backward, (iii) the dataflow equation for an assignment statement and a predicate, (iv) the confluence operator and (v) whether we need to compute the least or greatest solution.

   (a) A variable is said to be possibly uninitialized if there is a path in the control-flow graph from START to a use of that variable that does not pass through a definition of that variable. Compute the set of possibility uninitialized variables of a procedure.

   (b) A variable is said to be definitely uninitialized if all paths in the control-flow graph from START to a use of that variable do not pass through a definition of that variable. Compute the set of possibility uninitialized variables of a procedure.