Systems of Inequalities

Goals:

Given system of inequalities of the form $Ax \leq b$

- determine if system has an integer solution
- enumerate all integer solutions

Running example:

$$3x + 4y \geq 16 \tag{1}$$

$$4x + 7y \leq 56 \tag{2}$$

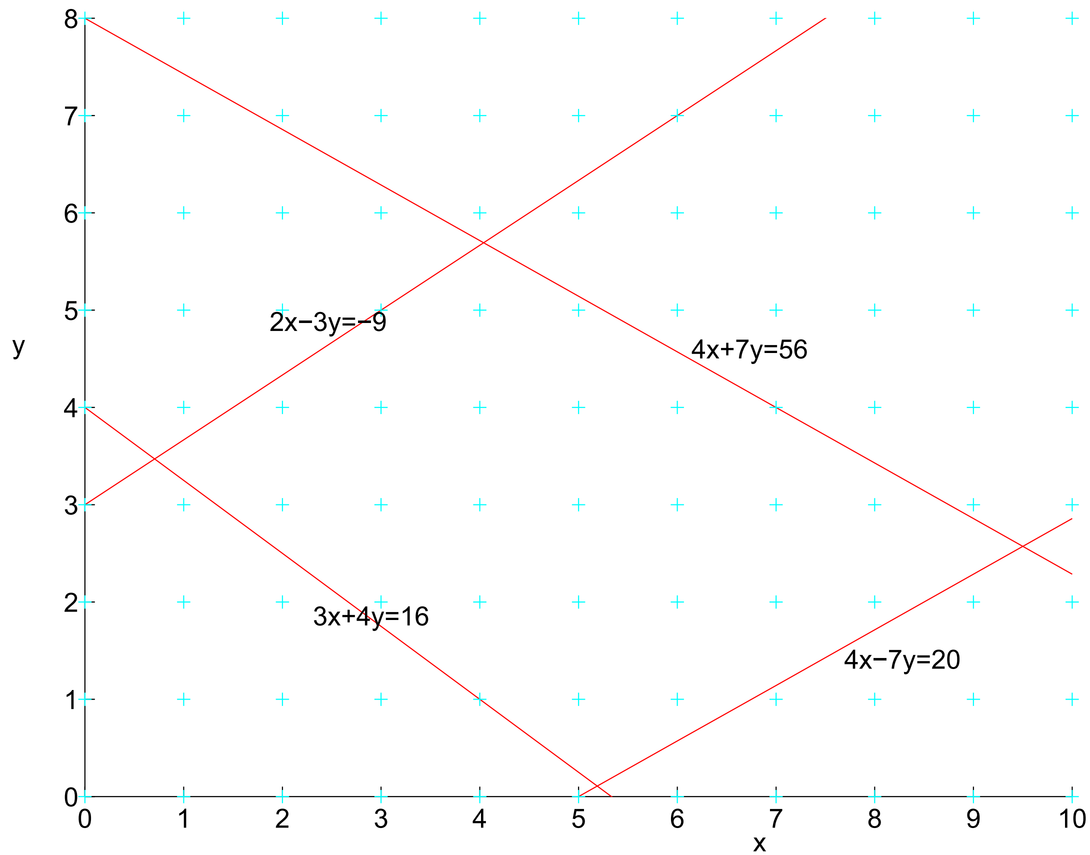
$$4x - 7y \leq 20 \tag{3}$$

$$2x - 3y \geq -9 \tag{4}$$

Upper bounds for x: (2) and (3)
Lower bounds for x: (1) and (4)

Upper bounds for y: (2) and (4)
Lower bounds for y: (1) and (3)

MATLAB graphs:



Code for enumerating integer points in polyhedron: (see graph)

Outer loop: Y, Inner loop: X

DO Y= $\lceil 4/37 \rceil$, $\lfloor 74/13 \rfloor$

  DO X= $\lceil max(16/3 - 4y/3, -9/2 + 3y/2) \rceil$, $\lfloor min(5 + 7y/4, 14 - 7y/4) \rfloor$

  .....

Outer loop: X, Inner loop: Y

DO X=1, 9

  DO Y= $\lceil max(4 - 3y/4, (4x - 20)/7) \rceil$, $\lfloor (min(8 - 4x/5, (2x + 9)/3) \rfloor$

  .....

How do we can determine loop bounds?

Fourier-Motzkin elimination: variable elimination technique for inequalities

$$3x + 4y \geq 16 \tag{5}$$

$$4x + 7y \leq 56 \tag{6}$$

$$4x - 7y \leq 20 \tag{7}$$

$$2x - 3y \geq -9 \tag{8}$$

Let us project out x.
First, express all inequalities as upper or lower bounds on x.

$$x \geq 16/3 - 4y/3 \tag{9}$$

$$x \leq 14 - 7y/4 \tag{10}$$

$$x \leq 5 + 7y/4 \tag{11}$$

$$x \geq -9/2 + 3y/2 \tag{12}$$

For any y, if there is an x that satisfies all inequalities, then every lower bound on x must be less than or equal to every upper bound on x.

Generate a new system of inequalities from each pair (upper,lower) bounds.

$$\begin{aligned}
5 + 7y/4 &\geq 16/3 - 4y/3 (\text{Inequalities} 3, 1) \\
5 + 7y/4 &\geq -9/2 + 3y/2 (\text{Inequalities} 3, 4) \\
14 - 7y/4 &\geq 16/3 - 4y/3 (\text{Inequalities} 2, 1) \\
14 - 7y/4 &\geq -9/2 + 3y/2 (\text{Inequalities} 2, 4)
\end{aligned}$$

Simplify:

$$
\begin{aligned}
y &\geq 4/37 \\
y &\geq -38 \\
y &\leq 104/5 \\
y &\leq 74/13
\end{aligned}
$$

=>

$$
\begin{aligned}
max(4/37, -38) &\leq y \leq min(104/5, 74/13) \\
&=> \\
4/37 &\leq y \leq 74/13
\end{aligned}
$$

This means there are rational solutions to original system of inequalities.

We can now express solutions in closed form as follows:

$$\begin{aligned} 4/37 &\leq y \leq 4/37 \\ max(16/3 - 4y/3, -9/2 + 3y/2) &\leq x \leq min(5 + 7y/4, 14 - 7y/4) \end{aligned}$$

Fourier-Motzkin elimination: iterative algorithm

Iterative step:

- obtain reduced system by projecting out a variable
- if reduced system has a rational solution, so does the original

Termination: no variables left

Projection along variable x: Divide inequalities into three categories

$$a_1 * y + a_2 * z + ... \leq c_1 (no\ x)$$
$$b_1 * x \leq c_2 + b_2 * y + b_3 * z + ...(upper\ bound)$$
$$d_1 * x \geq c_3 + d_2 * y + d_3 * z + ...(lower\ bound)$$

New system of inequalities:

- All inequalities that do not involve x
- Each pair (lower,upper) bounds gives rise to one inequality:

$$b_1[c_3 + d_2 * y + d_3 * z + ...] \leq d_1[c_2 + b_2 * y + b_3 * z + ...]$$

Theorem: If $(y_1, z_1, ...)$ satisfies the reduced system, then $(x_1, y_1, z_1...)$ satisfies the original system, where x_1 is a rational number between

$min(1/b_1(c_2 + b_2y_1 + b_3z_1 + ...), ...)$ (over all upper bounds)

and

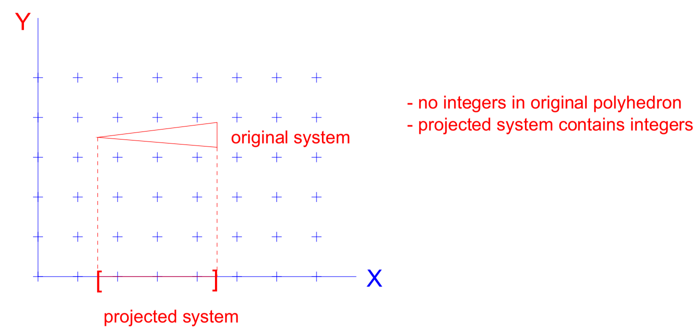
$max(1/d_1(c_3 + d_2y_1 + d_3z_1 + ...), ...)$ (over all lower bounds)

Proof: trivial

What can we conclude about integer solutions?

Corollary: If reduced system has no integer solutions, neither does the original system.

Not true: Reduced system has integer solutions => original system does too.



Key problem: Multiplying one inequality by b_1 and other by d_1 is not guaranteed to preserve "integrality" (cf. equalities)

Exact projection: If all upper bound coefficients b_i or all lower bound coefficients d_i happen to be 1, then integer solution to reduced system implies integer solution to original system.

Theorem: If $(y_1, z_1, ...)$ is an integer vector that satisfies the reduced system in FM elimination, then $(x_1, y_1, z_1...)$ satisfies the original system if there exists an integer x_1 between

$\lceil max(1/d_1(c_3 + d_2y_1 + d_3z_1 + ...), ...)\rceil$ (over all lower bounds)

and

$\lfloor min(1/b_1(c_2 + b_2y_1 + b_3z_1 + ...), ...)\rfloor$ (over all upper bounds).

Proof: trivial

Enumeration: Given a system $Ax \leq b$, we can use Fourier-Motzkin elimination to generate a loop nest to enumerate all integer points that satisfy system as follows:

- pick an order to eliminate variables (this will be the order of variables from innermost loop to outermost loop)
- eliminate variables in that order to generate upper and lower bounds for loops as shown in theorem in previous slide

Remark: if polyhedron has no integer points, then the lower bound of some loop in the loop nest will be bigger than the upper bound of that loop

Existence: Given a system $Ax \leq b$, we can use Fourier-Motzkin elimination to project down to a single variable.

- If the reduced system has no integer solutions, then original system has no integer solutions either.
- If the reduced system has integer solutions and all projections were exact, then original system has integer solutions too.
- If reduced system has integer solutions and some projections were no exact, be conservative and assume that original system has integer solutions.