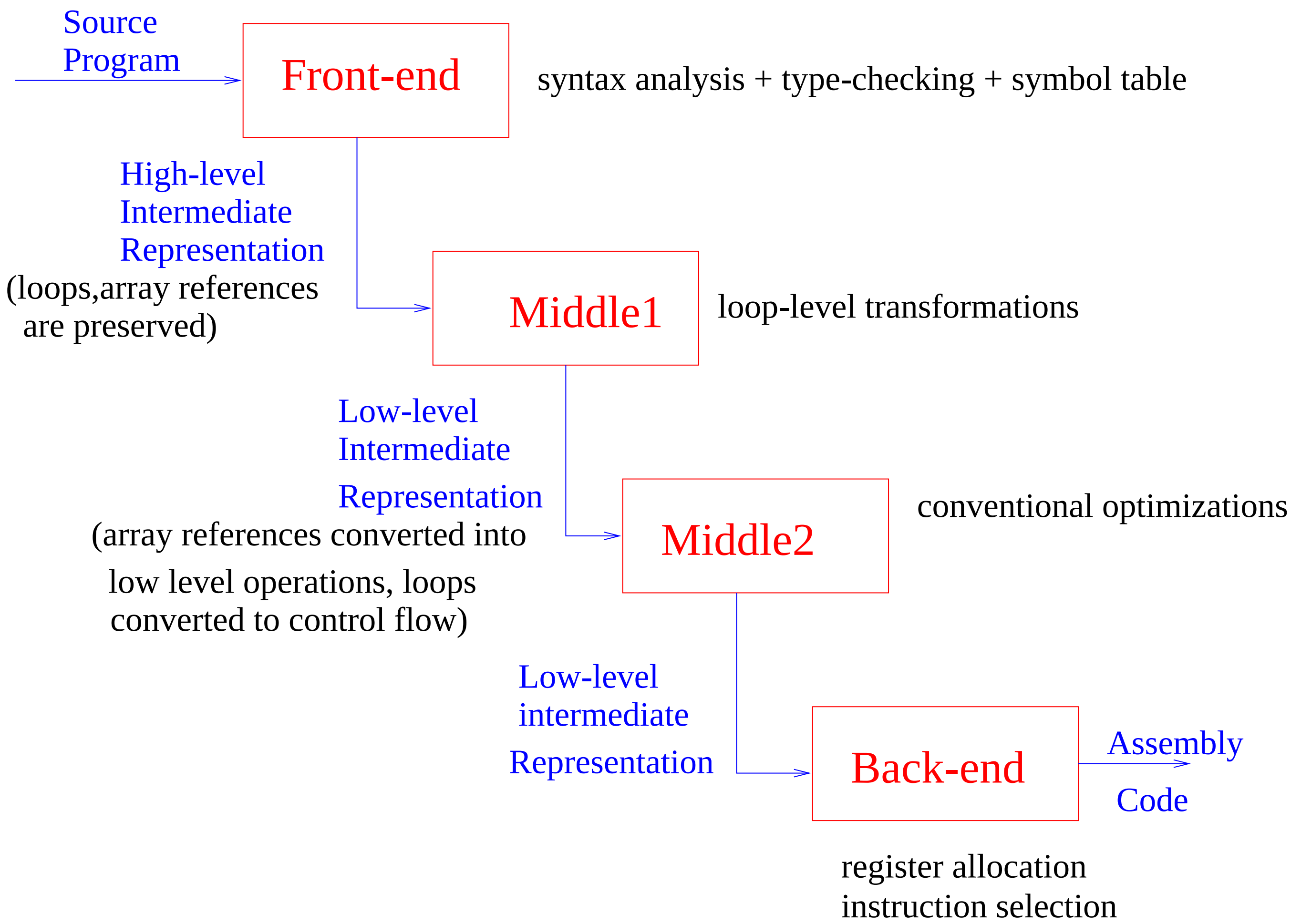
Introduction
to
Loop Transformations

Organization of a Modern Compiler



Key concepts:

Perfectly-nested loop: Loop nest in which all assignment statements occur in body of innermost loop.

```
for J = 1, N
  for I = 1, N
    Y(I) = Y(I) + A(I,J)*X(J)
```

Imperfectly-nested loop: Loop nest in which some assignment statements occur within some but not all loops of loop nest

```
  for k = 1, N
    a(k,k) = sqrt (a(k,k))
    for i = k+1, N
      a(i,k) = a(i,k) / a(k,k)
    for i = k+1, N
      for j = k+1, i
        a(i,j) -= a(i,k) * a(j,k)
```

Our focus for now: perfectly-nested loops

Goal of lecture:

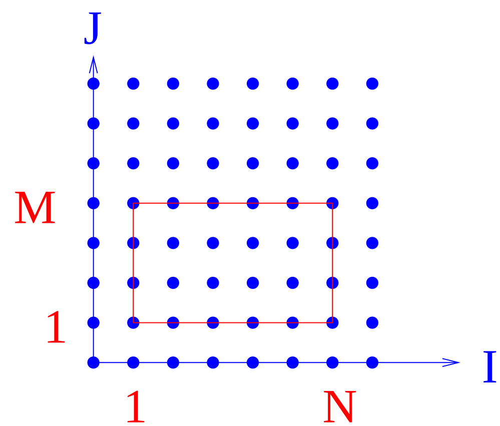
- We have seen two key transformations of perfectly-nested loops for locality enhancement: permutation and tiling.
- There are other loop transformations that we will discuss in class.
- Powerful way of thinking of perfectly-nested loop execution and transformations:
  - loop body instances $\leftrightarrow$ *iteration space* of loop
  - loop transformation $\leftrightarrow$ *change of basis* for iteration space

Iteration Space of a Perfectly-nested Loop

Each iteration of a loop nest with n loops can be viewed as an integer point in an n-dimensional space.

Iteration space of loop: all points in n-dimensional space corresponding to loop iterations

```
DO  I = 1, N
  DO  J  = 1,M
     S
```



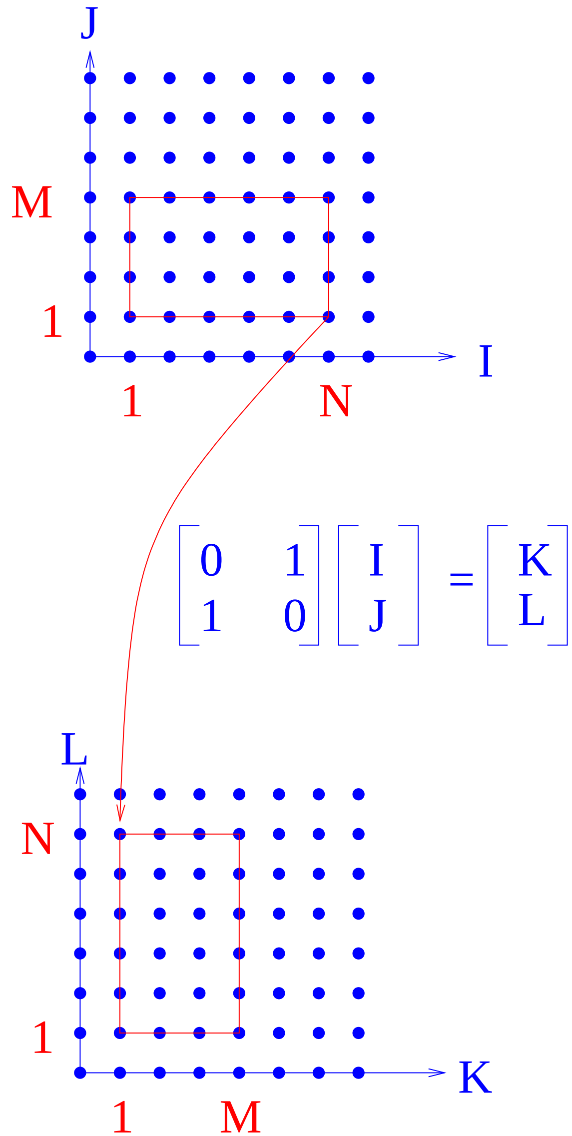
Execution order = lexicographic order on iteration space:

$$(1,1) \preceq (1,2) \preceq ... \preceq (1,M) \preceq (2,1) \preceq (2,2) ... \preceq (N,M)$$

Loop permutation = linear transformation on iteration space

```
DO  I = 1, N
  DO  J  = 1,M
     S(I,J)
```

```
DO  K  = 1, M
  DO  L = 1, N
    S'(K,L)
```



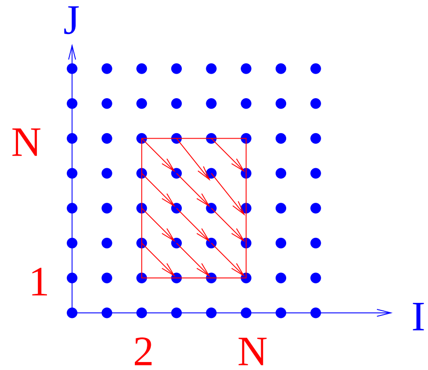
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ J \end{bmatrix} = \begin{bmatrix} K \\ L \end{bmatrix}$$

Locality enhancement:

Loop permutation brings iterations that touch the same cache line "closer" together, so probability of cache hits is increased.

Subtle issue 1: loop permutation may be illegal in some loop nests

```
DO  I = 2, N
 DO  J  = 1, M
   A[I,J] = A[I-1,J+1] + 1
```



Assume that array has 1's stored everywhere before loop begins.

After loop permutation:

```
DO  J  = 1, M
  DO  I = 2, N
    A[I,J] = A[I-1,J+1] + 1
```

Transformed loop will produce different values (A[3,1] for example)
=> permutation is illegal for this loop.

Question: How do we determine when loop permutation is legal?

Subtle issue 2: generating code for transformed loop nest may be non-trivial!

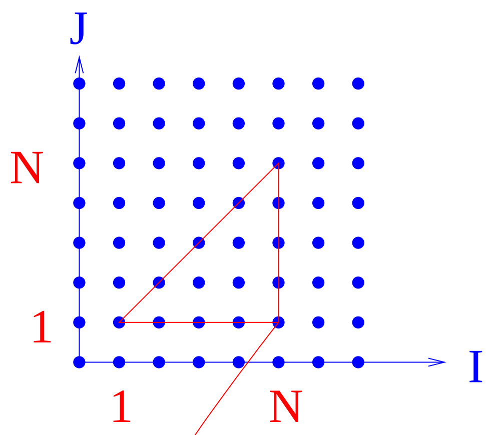
Example: triangular loop bounds (triangular solve/Cholesky)

```
FOR I = 1, N
  FOR J = 1, I-1
     S
```

Here, inner loop bounds are functions of outer loop indices!

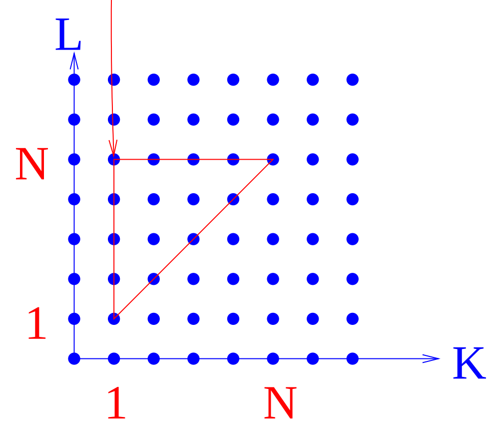
Just exchanging the two loops will not generate correct bounds.

```
DO  I = 1, N
  DO  J  = 1, I
     S(I,J)
```



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ J \end{bmatrix} = \begin{bmatrix} K \\ L \end{bmatrix}$$

```
DO  K  = 1, N
 DO  L = K, N
    S'(K,L)
```



Question: How do we generate loop bounds for transformed loop nest?

General theory of oop transformations shou d te   us

- which transformations are lega ,
- what the best sequence of transformations should be for a given target architecture, and
- what the transformed code shou d be.

Desirable: quantitative estimates of performance improvement

ILP Formulation of Loop Transformations

Goal:

1. formu ate correctness of permutation as integer linear programming (ILP) problem
2. formulate code generation problem as ILP

Two problems:

Given a system of linear inequalities $A\,x \leq b$

where A is a m X n matrix of integers,

b is an m vector of integers,

x is an n vector of unknowns,

(i) Are there integer solutions?

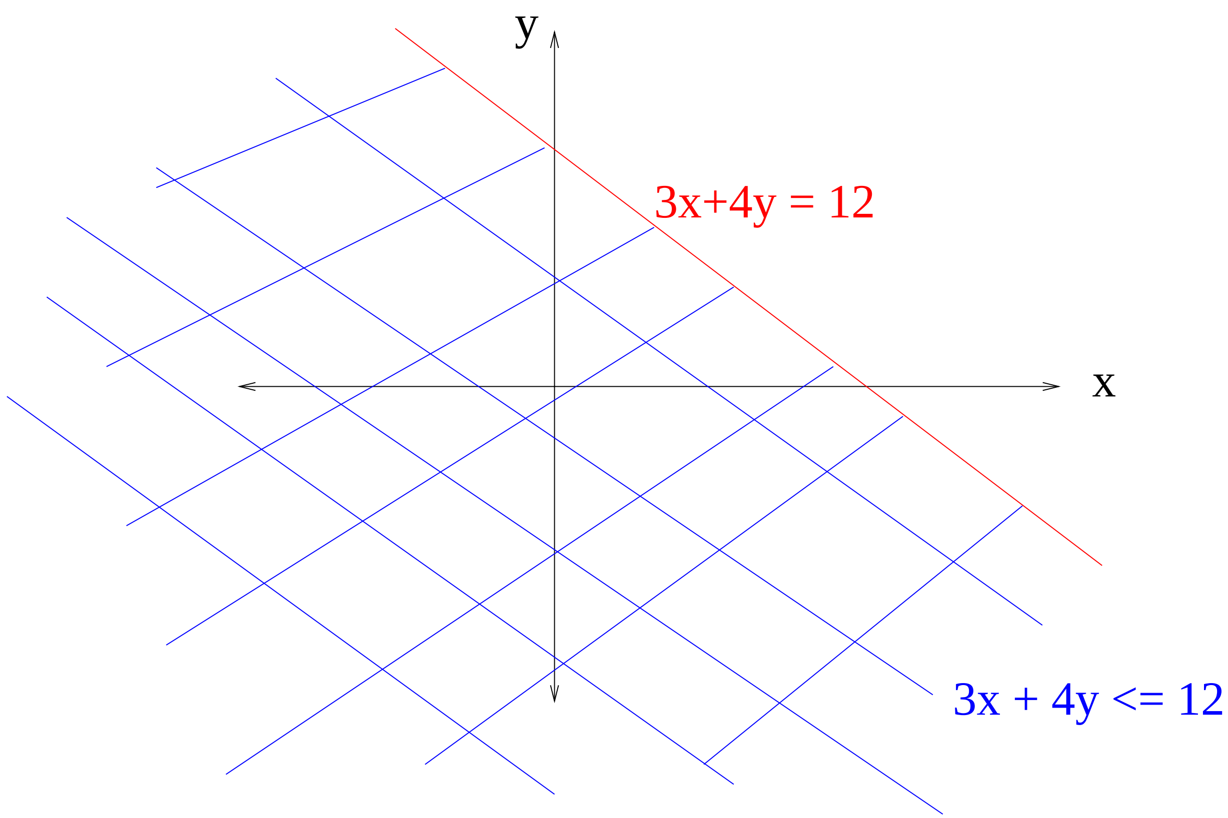
(ii) Enumerate all integer solutions.

Most problems regarding correctness of transformations and code generation can be reduced to these problems.

Intuition about systems of linear inequalities:

Equality: line (2D), plane (3D), hyperplane (> 3D)

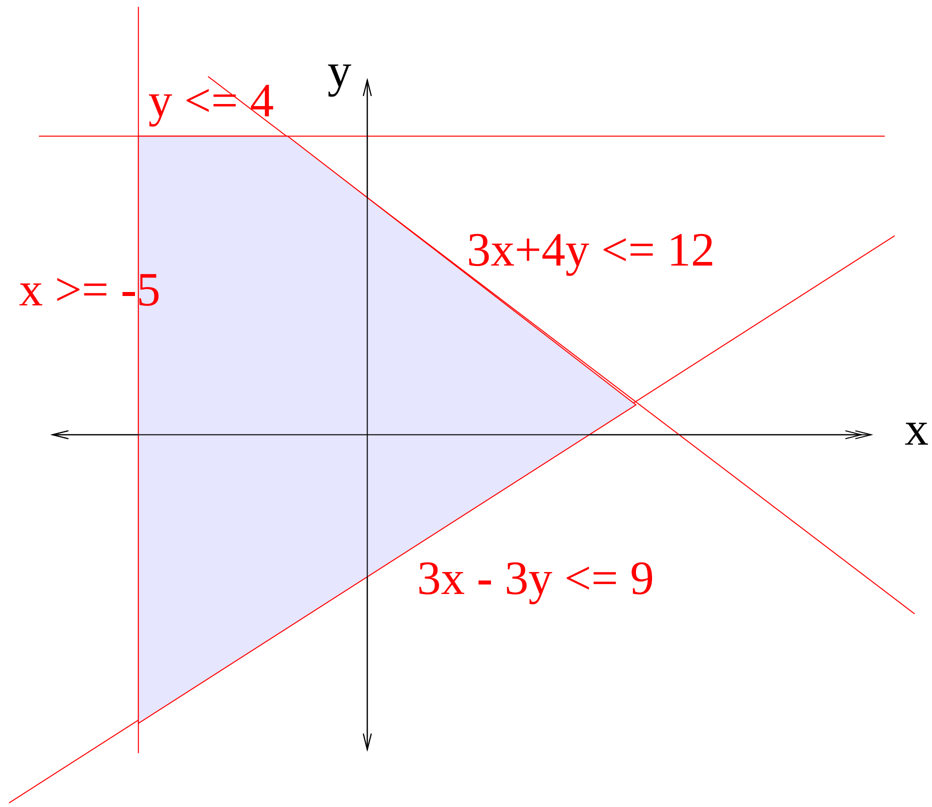
Inequality: half-plane (2D), half-space(>2D)



Region described by inequality is convex
(if two points are in region, all points in between them are in region)

Intuition about systems of linear inequalities:

Conjunction of inequalties = intersection of half-spaces
=> some convex region



Region described by inequalities is a convex polyhedron
(if two points are in region, all points in between them are in region)

Let us formulate correctness of loop permutation as ILP problem.

Intuition: If a iterations of a loop nest are independent, then permutation is certainly legal.

This is stronger than we need, but it is a good starting point.

What does independent mean?

Let us ook at dependences.

Dependences:
- data
  - flow
  - anti
  - output
- control

Flow dependence: S1 -> S2

(i) S1 executes before S2 in program order

(ii) S1 writes into a location that is read by S2

Anti-dependence: S1 -> S2

(i) S1 executes before S2

(ii) S1 reads from a location that is overwritten later by S2

Output dependence: S1 -> S2

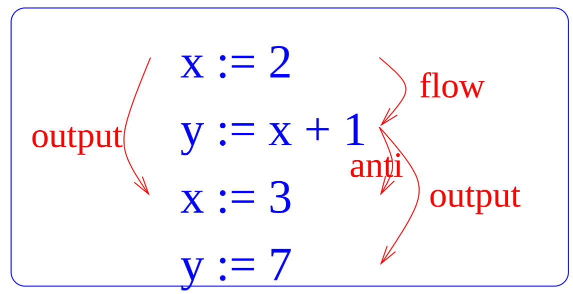
(i) S1 executes before S2

(ii) S1 and S2 write to the same location

Input dependence: S1 -> S2

(i) S1 executes before S2

(ii) S1 and S2 both read from the same location



Input dependence is not usually important for most app ications.

Conservative Approximation:

- Real programs: imprecise information => need for safe approximation

  'When you are not sure whether a dependence exists, you must assume it does.'

Example:

```
procedure f (X,i,j)
  begin
    X(i) = 10;
    X(j) = 5;
  end
```

Question: Is there an output dependence from the first assignment to the second?

Answer: If (i = j), there is a dependence; otherwise, not.

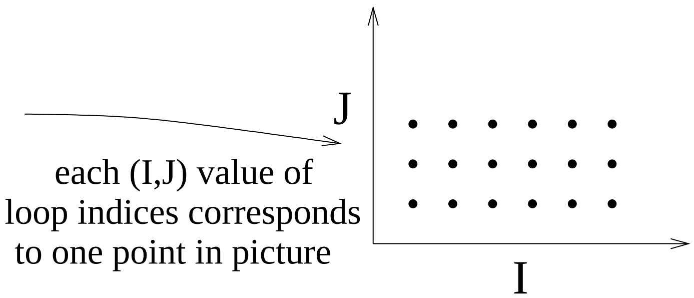
=> Unless we know from interprocedural analysis that the parameters i and j are always distinct, we must play it safe and insert the dependence.

Key notion: Aliasing : two program names may refer to the same location (like X(i) and X(j))

May-dependence vs must-dependence: More precise analysis may eliminate may-dependences

Loop level Analysis: granularity is a loop iteration

```
DO I = 1, 100
  DO J = 1, 100
    S
```



Dynamic instance of a statement:

Execution of a statement for given loop index values

Dependence between iterations:

Iteration (I1,J1) is said to be dependent on iteration (I2,J2) if a dynamic instance (I1,J1) of a statement in loop body is dependent on a dynamic instance (I2,J2) of a statement in the loop body.

How do we compute dependences between iterations of a loop nest?

Dependences in loops

```
      FOR 10 I = 1, N
        X(f(I)) = ...
     10         = ...X(g(I))..
```

- Conditions for flow dependence from iteration I_w to I_r:
  - $1 \leq I_w \leq I_r \leq N$ *(write before read)*
  - $f(I_w) = g(I_r)$ *(same array location)*
- Conditions for anti-dependence from iteration I_g to I_o:
  - $1 \leq I_g < I_o \leq N$ *(read before write)*
  - $f(I_o) = g(I_g)$ *(same array location)*
- Conditions for output dependence from iteration I_{w1} to I_{w2}:
  - $1 \leq I_{w1} < I_{w2} \leq N$ *(write in program order)*
  - $f(I_{w1}) = f(I_{w2})$ *(same array location)*

Dependences in nested loops

```
     FOR 10 I = 1, 100
      FOR 10 J = 1, 200
       X(f(I,J),g(I,J)) = ...
10          = ...X(h(I,J),k(I,J))..
```

Conditions for flow dependence from iteration (I_w, J_w) to (I_r, J_r):
Recall: $\preceq$ is the lexicographic order on iterations of nested loops.

$$
\begin{aligned}
1 &\leq I_w \leq 100 \\
1 &\leq J_w \leq 200 \\
1 &\leq I_r \leq 100 \\
1 &\leq J_r \leq 200 \\
(I_w, J_w) &\preceq (I_r, J_r) \\
f(I_w, J_w) &= h(I_r, J_r) \\
g(I_w, J_w) &= k(I_r, J_r)
\end{aligned}
$$

Anti and output dependences can be defined analogously.

Array subscripts are affine functions of loop variables

=>

dependence testing can be formulated as a set of ILP problems

ILP Formulation

```
FOR I = 1, 100
  X(2I) = .... X(2I+1)...
```

Is there a flow dependence between different iterations?

$$\begin{aligned} 1 &\leq Iw < Ir \leq 100 \\ 2Iw &= 2Ir + 1 \end{aligned}$$

which can be written as

$$\begin{aligned} 1 &\leq Iw \\ Iw &\leq Ir - 1 \\ Ir &\leq 100 \\ 2Iw &\leq 2Ir + 1 \\ 2Ir + 1 &\leq 2Iw \end{aligned}$$

The system

$$\begin{aligned} 1 &\leq Iw \\ Iw &\leq Ir - 1 \\ Ir &\leq 100 \\ 2Iw &\leq 2Ir + 1 \\ 2Ir + 1 &\leq 2Iw \end{aligned}$$

can be expressed in the form $Ax \leq b$ as follows

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{bmatrix} Iw \\ Ir \end{bmatrix} \leq \begin{bmatrix} -1 \\ -1 \\ 100 \\ 1 \\ -1 \end{bmatrix}$$

ILP Formulation for Nested Loops

```
FOR I = 1, 100
 FOR J = 1, 100
   X(I,J) = ..X(I-1,J+1)...
```

Is there a flow dependence between different iterations?

$$
\begin{aligned}
1 &\leq Iw \leq 100 \\
1 &\leq Ir \leq 100 \\
1 &\leq Jw \leq 100 \\
1 &\leq Jr \leq 100 \\
(Iw, Jw) &\prec (Ir, Jr)(lexicographic\ order) \\
Ir - 1 &= Iw \\
Jr + 1 &= Jw
\end{aligned}
$$

Convert lexicographic order $\prec$ into integer equalities/inequalities.

$(Iw, Jw) \prec (Ir, Jr)$ is equivalent to

$Iw < Ir$ OR $((Iw = Ir)\ AND\ (Jw < Jr))$

We end up with two systems of inequalities:

$$
\begin{array}{l}
1 \le Iw \le 100 \\
1 \le Ir \le 100 \\
1 \le Jw \le 100 \\
1 \le Jr \le 100 \\
Iw < Ir \\
Ir - 1 = Iw \\
Jr + 1 = Jw
\end{array}
\qquad OR \qquad
\begin{array}{l}
1 \le Iw \le 100 \\
1 \le Ir \le 100 \\
1 \le Jw \le 100 \\
1 \le Jr \le 100 \\
Iw = Ir \\
Jw < Jr \\
Ir - 1 = Iw \\
Jr + 1 = Jw
\end{array}
$$

Dependence exists if either system has a solution.

What about affine loop bounds?

```
FOR I = 1, 100
  FOR J = 1, I
    X(I,J) = ..X(I-1,J+1)...
```

$$
\begin{aligned}
1 &\leq Iw \leq 100 \\
1 &\leq Ir \leq 100 \\
1 &\leq Jw \leq Iw \\
1 &\leq Jr \leq Ir \\
(Iw, Jw) &\prec (Ir, Jr)(lexicographic order) \\
Ir - 1 &= Iw \\
Jr + 1 &= Jw
\end{aligned}
$$

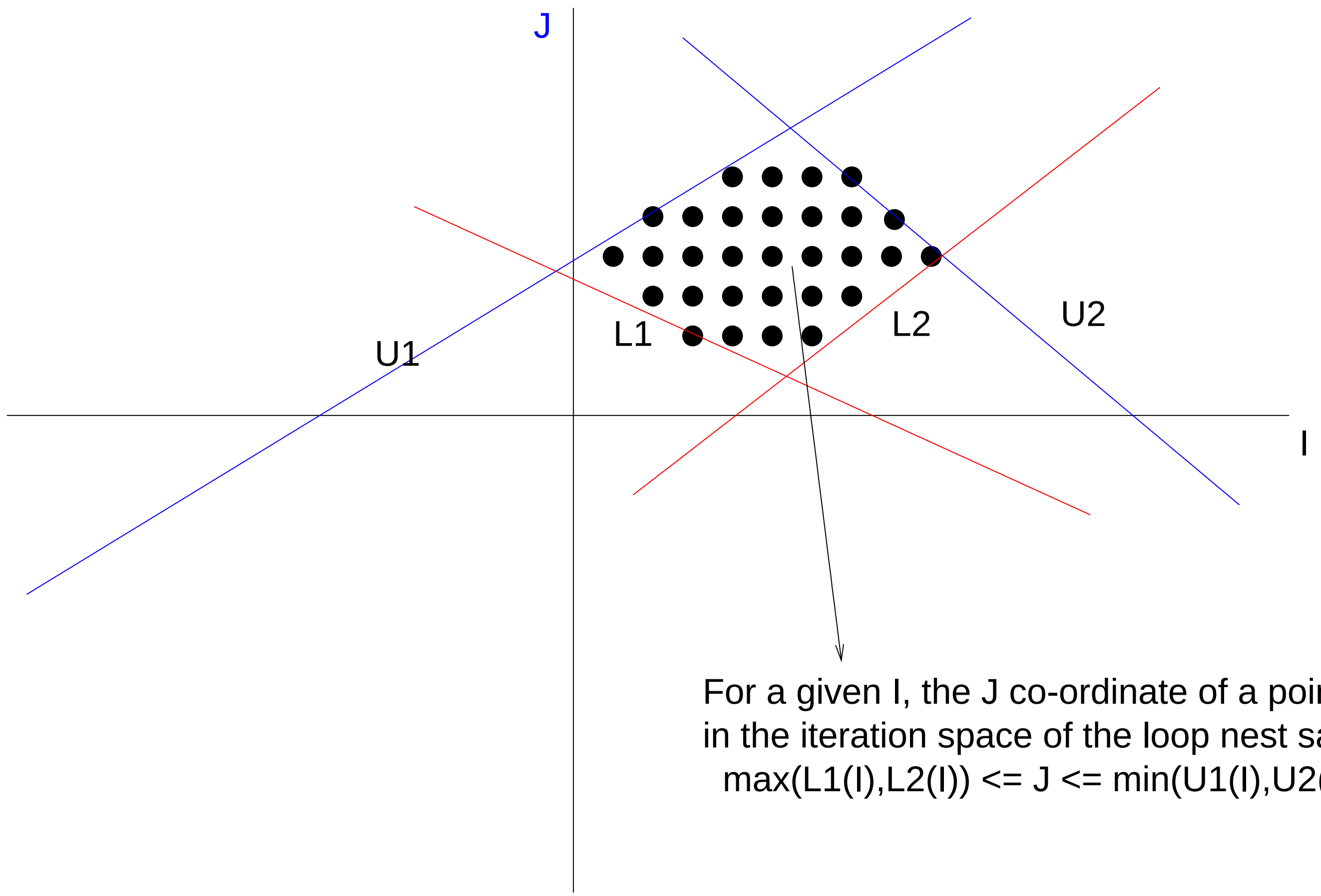
We can actually handle fairly complicated bounds involving min's and max's.

```
FOR I = 1, 100
  FOR J = max(F1(I),F2(I)) ,  min(G1(I),G2(I))
     X(I,J) = ..X(I-1,J+1)...
```

$$
\begin{aligned}
& \ldots. \\
F1(Ir) &\leq Jr \\
F2(Ir) &\leq Jr \\
Jr &\leq G1(Ir) \\
Jr &\leq G2(Ir) \\
& \ldots.
\end{aligned}
$$

Caveat: $F1, F2$ etc. must be affine functions.

Min’s and max’s in loop bounds mayseem weird, but actually they describe general polyhedral iteration spaces!



More important case in practice: variables in upper/lower bounds

```
FOR I = 1, N
  FOR J = 1 , N-1
    ....
```

Solution: Treat N as though it was an unknown in system

$$
\begin{aligned}
1 &\leq Iw \leq N \\
1 &\leq Jw \leq N-1 \\
\ldots&.
\end{aligned}
$$

This is equivalent to seeing if there is a solution for any value of N.

Note: if we have more information about the range of N, we can easily add it as additional inequalities.

Summary

Problem of determining if a dependence exists between two iterations of a perfectly nested loop can be framed as ILP problem of the form

Is there an integer solution to system $Ax \leq b$?

How do we solve this decision problem?

Is there an integer solution to system $Ax \leq b$?

Oldest solution technique: Fourier-Motzkin elimination

Intuition: ”Gaussian elimination for inequalties”

More modern techniques exist, but all known solutions require time exponential in the number of inequalities

=>

Anything you can do to reduce the number of inequalities is good.

=>

Equalities should not be converted blindly into inequalities but handled separately.

Presentation sequence:

- one equation, several variables

  $2x + 3y = 5$

- several equations, several variables

  $$\begin{aligned} 2x + 3y + 5z &= 5 \\ 3x + 4y &= 3 \end{aligned}$$

- equations & inequalities

  $$\begin{aligned} 2x + 3y &= 5 \\ x &<= 5 \\ y &<= -9 \end{aligned}$$

Diophatine equations: use integer Gaussian elimination

Solve equalities first then use Fourier-Motzkin elimination

One equation, many variables:

Thm: The linear Diophatine equation a1 x1 + a2 x2 + ...+ an xn = c has integer solutions iff gcd(a1,a2,...,an) divides c.

Examples:

(1) $2x = 3$ No solutions

(2) $2x = 6$ One solution: $x = 3$

(3) $2x + y = 3$

GCD(2,1) = 1 which divides 3.

Solutions: $x = t$, $y = (3 - 2t)$

(4) $2x + 3y = 3$

GCD(2,3) = 1 which divides 3.

Let $z = x + \mathrm{floor}(3/2)\, y = x + y$

Rewrite equation as $2z + y = 3$

Solutions: $z = t$, $y = (3 - 2t)$ => $x = (3t - 3)$, $y = (3 - 2t)$

Intuition: Think of underdetermined systems of eqns over reals.

Caution: Integer constraint => Diophantine system may have no solns

Thm: The linear Diophatine equation $a1\ x1 + a2\ x2 + \ldots + an\ xn = c$ has integer solutions iff gcd(a1,a2,...,an) divides c.

Proof: WLOG, assume that all coefficients a1,a2,...an are positive.
We prove only the IF case by induction, the proof in the other direction is trivial.
Induction is on min(smallest coefficient, number of variables).

Base case:

If (# of variables = 1) , then equation is a1 x1 = c which has integer solutions if a1 divides c.

If (smallest coefficient = 1), then gcd(a1,a2,...,an) = 1 which divides c.

Wlog, assume that a1 = 1, and observe that the equation has solutions of the form (c - a2 t2 - a3 t3 -....-an tn, t2, t3, ...tn).

Inductive case:

Suppose smallest coefficient is a1, and let $t = x1 + \text{floor}(a2/a1)\ x2 + \ldots + \text{floor}(an/a1)\ xn$

In terms of this variable, the equation can be rewritten as

$(a1)\ t + (a2 \bmod a1)\ x2 + \ldots + (an \bmod a1)\ xn = c$ (1)

where we assume that all terms with zero coefficient have been deleted.

Observe that (1) has integer solutions iff original equation does too.

Now gcd(a,b) = gcd(a mod b, b) => gcd(a1,a2,...,an) = gcd(a1, (a2 mod a1),..,(an mod a1))
=> gcd(a1, (a2 mod a1),..,(an mod a1)) divides c.

If a1 is the smallest co-efficient in (1), we are left with 1 variable base case.

Otherwise, the size of the smallest co-efficient has decreased, so we have made progress in the induction.

Summary:

$\boxed{\text{Eqn: } a_1 x_1 + a_2 x_2 + \dots + a_n x_n = c}$

- Does this have integer solutions?

= Does gcd(a1,a2,...,an) divide c ?

It is useful to consider solution process in matrix-theoretic terms.

We can write single equation as

$$(3\ 5\ 8)(x\ y\ z)^T = 6$$

It is hard to read off solution from this, but for special matrices, it is easy.

$$(2\ \ 0)(a\ \ b)^T = 8$$

Solution is a = 4, b = t

looks lower triangular, right?

Key concept: column echelon form -

"lower triangular form for underdetermined systems"

For a matrix with a single row, column echelon form is

$(x\ \ 0\ \ 0\ \ 0...0)$

$3x + 5y + 8z = 6$

Substitution: $t = x + y + 2z$
New equation:

$3t + 2y + 2z = 6$

Substitution: $u = y+z+t$
New equation:

$2u + t = 6$

Solution:
$u = p1$
$t = (6-2p1)$

Backsubstitution:
$y = p2$
$t = (6-2p1)$
$z = (3p1-p2-6)$

Backsubstitution:
$x = (18-8p1+p2)$
$y = p2$
$z = (3p1-p2-6)$

$(3\ 5\ 8)$

$$(3\ 5\ 8)\begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{U1}$$

$= (3\ 2\ 2)$

$$(3\ 2\ 2)\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{U2}$$

$= (1\ \ 2\ \ 0)$

$$(1\ \ 2\ \ 0)\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{U3}$$

$= (1\ 0\ \ 0)$

Solution: $(6\ \ a\ \ b)^T$

U1*U2*U3

Product of matrices = $\begin{pmatrix} 2 & -5 & -1 \\ -1 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution to original system: $\begin{pmatrix} 12-5a-b \\ -6+3a-b \\ b \end{pmatrix}$

U1*U2*U3*$(6\ a\ b)^T$

Systems of Diophatine Equations:

Key idea: use integer Gaussian elimination

Example:

$$\begin{aligned} 2x + 3y + 4z &= 5 \\ x - \ \ y + 2z &= 5 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

It is not easy to determine if this Diophatine system has solutions.

Easy special case: lower triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x &= 5 \\ y &= 3 \\ z &= \text{arbitrary integer} \end{aligned}$$

Question: Can we convert general integer matrix into equivalent lower triangular system?

INTEGER GAUSSIAN ELIMINATION

Integer gaussian Elimination

- Use row/column operations to get matrix into triangular form
- For us, column operations are more important because we usually have more unknowns than equations

Overall strategy: Given Ax = b

Find matrices U1, U2,...Uk such that

A*U1*U2*...*Uk is lower triangular (say L)
Solve Lx' = b (easy)
Compute x = (U1*U2*...*Uk)*x

Proof:

(A*U1*U2...*Uk)x' = b
=> A(U1*U2*...*Uk)x' = b
=> x = (U1*U2...*Uk)x'

Caution: Not all column operations preserve integer solutions.

$$\begin{bmatrix}2 & 3\\ 6 & 7\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}5\\ 1\end{bmatrix}$$

Solution: $x = -8$, $y = 7$

$$\Big\downarrow \begin{bmatrix}1 & -3\\ 0 & 2\end{bmatrix}$$

$$\begin{bmatrix}2 & 0\\ 6 & -4\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}5\\ 1\end{bmatrix}$$

which has no integer solutions!

Intuition: With some column operations, recovering solution of original system requires solving lower triangular system using rationals.

Question: Can we stay purely in the integer domain?

One solution: Use only unimodular column operations

Unimodular Column Operations:

(a) Interchange two columns

$$\begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \xrightarrow[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}]{} \begin{bmatrix} 3 & 2 \\ 7 & 6 \end{bmatrix}$$

Check

Let x,y satisfy first eqn.

Let x',y' satisfy second eqn.

$x' = y \,, \quad y' = x$

(b) Negate a column

$$\begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}]{} \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$$

Check

$x' = x, \quad y' = -y$

(c) Add an integer multiple of one column to another

$$\begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}]{} \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}$$

n = -1

Check

$x = x' + n\,y'$

$y = y'$

Example:

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 5 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \Rightarrow \begin{matrix} x' = 5 \\ y' = 3 \\ z' = t \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ t \end{bmatrix} = \begin{bmatrix} 4-2t \\ -1 \\ t \end{bmatrix}$$

Facts:

1. The three unimodular column operations
   - interchanging two columns
   - negating a column
   - adding an integer multiple of one column to another

   on the matrix A of the system A x = b
   preserve integer solutions, as do sequences of these operations.

2. Unimodular column operations can be used to reduce a matrix A into lower triangular form.

3. A unimodular matrix has integer entries and a determinant of +1 or -1.

4. The product of two unimodular matrices is also unimodular.

Algorithm: Given a system of Diophantine equations Ax = b

1. Use unimodular column operations to reduce matrix A to lower triangular form L.
2. If Lx' = b has integer solutions, so does the original system.
3. If explicit form of solutions is desired, let U be the product of unimodular matrices corresponding to the column operations.

   x = U x' where x' is the solution of the system Lx' = b

Detail: Instead of lower triangular matrix, you should to compute 'column echelon form' of matrix.

Column echelon form: Let rj be the row containing the first non-zero in column j.

(i) r(j+1) > rj if column j is not entirely zero.

(ii) column (j+1) is zero if column j is.

$$\begin{bmatrix} x & 0 & 0 \\ x & 0 & 0 \\ x & x & x \end{bmatrix}$$

is lower triangular but not column echelon.
Point: writing down the solution for this system requires additional work with the last equation (1 equation, 2 variables). This work is precisely what is required to produce the column echelon form.

Note: Even in regular Gaussian elimination, we want column echelon form rather than lower triangular form when we have under-determined systems.

Systems of Inequalities

Goals:

Given system of inequalities of the form $Ax \leq b$

- determine if system has an integer solution
- enumerate all integer solutions

Running example:

$$3x + 4y \geq 16 \tag{1}$$

$$4x + 7y \leq 56 \tag{2}$$

$$4x - 7y \leq 20 \tag{3}$$

$$2x - 3y \geq -9 \tag{4}$$

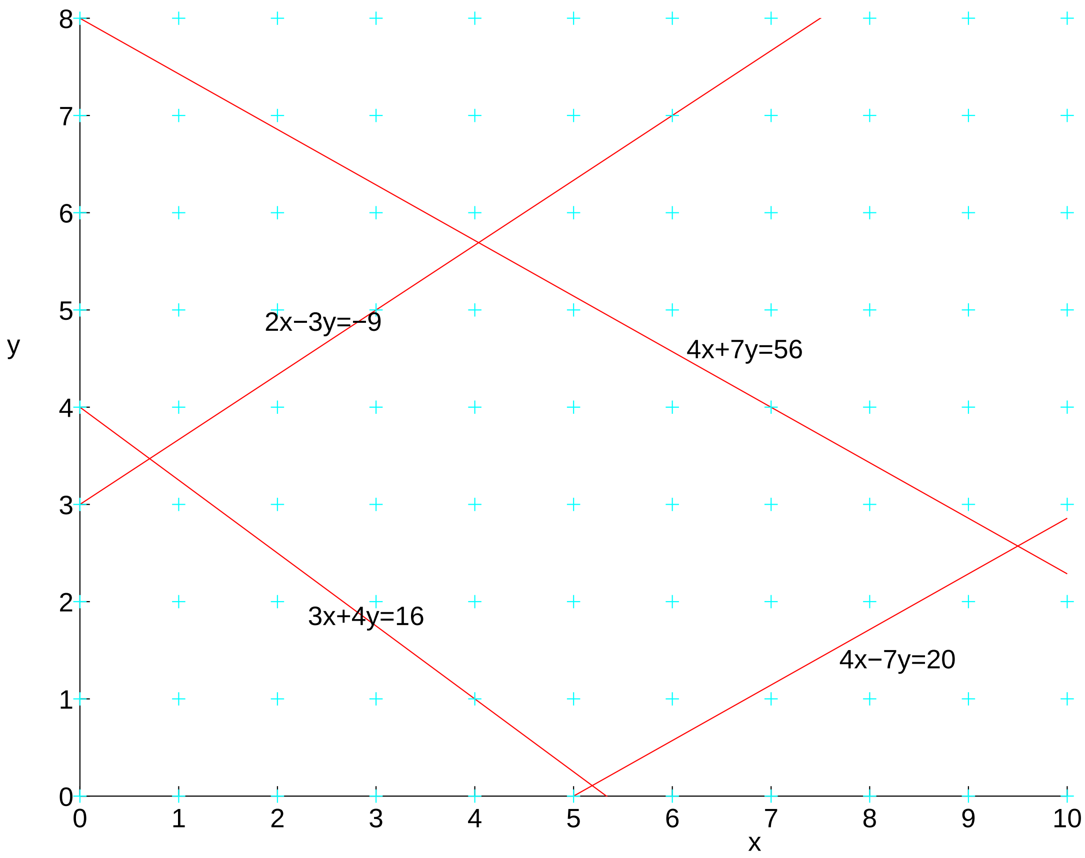
Upper bounds for x: (2) and (3)

Lower bounds for x: (1) and (4)

Upper bounds for y: (2) and (4)

Lower bounds for y: (1) and (3)

MATLAB graphs:



Code for enumerating integer points in polyhedron: (see graph)

Outer loop: Y, Inner loop: X

DO Y=$\lceil 4/37 \rceil$,$\lfloor 74/13 \rfloor$

DO X=$\lceil max(16/3 - 4y/3, -9/2 + 3y/2) \rceil$,$\lfloor min(5 + 7y/4, 14 - 7y/4) \rfloor$

. . . . .

Outer loop: X, Inner loop: Y

DO X=1, 9

DO Y=$\lceil max(4 - 3y/4, (4x - 20)/7) \rceil$,$\lfloor (min(8 - 4x/5, (2x + 9)/3) \rfloor$

. . . . .

How do we can determine loop bounds?

Fourier-Motzkin elimination: variable elimination technique for inequalities

$$3x + 4y \geq 16 \tag{5}$$

$$4x + 7y \leq 56 \tag{6}$$

$$4x - 7y \leq 20 \tag{7}$$

$$2x - 3y \geq -9 \tag{8}$$

Let us project out x.
First, express all inequalities as upper or lower bounds on x.

$$x \geq 16/3 - 4y/3 \tag{9}$$

$$x \leq 14 - 7y/4 \tag{10}$$

$$x \leq 5 + 7y/4 \tag{11}$$

$$x \geq -9/2 + 3y/2 \tag{12}$$

For any y, if there is an x that satisfies all inequalities, then every lower bound on x must be less than or equal to every upper bound on x.

Generate a new system of inequalities from each pair (upper,lower) bounds.

$$\begin{aligned} 5 + 7y/4 &\geq 16/3 - 4y/3 (\text{Inequalities} 3, 1) \\ 5 + 7y/4 &\geq -9/2 + 3y/2 (\text{Inequalities} 3, 4) \\ 14 - 7y/4 &\geq 16/3 - 4y/3 (\text{Inequalities} 2, 1) \\ 14 - 7y/4 &\geq -9/2 + 3y/2 (\text{Inequalities} 2, 4) \end{aligned}$$

Simplify:

$$
\begin{aligned}
y &\geq 4/37 \\
y &\geq -38 \\
y &\leq 104/5 \\
y &\leq 74/13
\end{aligned}
$$

=>

$$
\begin{aligned}
max(4/37, -38) \quad &\leq \quad y \leq min(104/5, 74/13) \\
&=> \\
4/37 \quad &\leq y \quad \leq 74/13
\end{aligned}
$$

This means there are rational solutions to original system of inequalities.

We can now express solutions in closed form as follows:

$$
\begin{aligned}
4/37 &\leq y \leq 4/37 \\
max(16/3 - 4y/3, -9/2 + 3y/2) &\leq x \leq min(5 + 7y/4, 14 - 7y/4)
\end{aligned}
$$

Fourier-Motzkin elimination: iterative algorithm

Iterative step:

- obtain reduced system by projecting out a variable
- if reduced system has a rational solution, so does the original

Termination: no variables left

Projection along variable x: Divide inequalities into three categories

$$a_1 * y + a_2 * z + ... \leq c_1 (no\ x)$$
$$b_1 * x \leq c_2 + b_2 * y + b_3 * z + ...(upper\ bound)$$
$$d_1 * x \geq c_3 + d_2 * y + d_3 * z + ...(lower\ bound)$$

New system of inequalities:

- All inequalities that do not involve x
- Each pair (lower,upper) bounds gives rise to one inequality:

$$b_1[c_3 + d_2 * y + d_3 * z + ...] \leq d_1[c_2 + b_2 * y + b_3 * z + ...]$$

Theorem: If $(y_1, z_1, ...)$ satisfies the reduced system, then $(x_1, y_1, z_1 ...)$ satisfies the original system, where x_1 is a rational number between

$min(1/b_1(c_2 + b_2 y_1 + b_3 z_1 + ...), ...)$ (over all upper bounds)

and

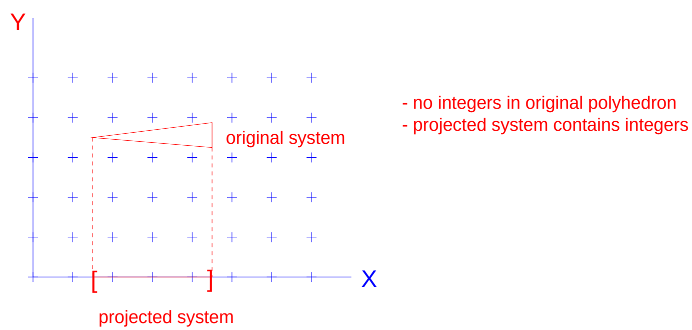
$max(1/d_1(c_3 + d_2 y_1 + d_3 z_1 + ...), ...)$ (over all lower bounds)

Proof: trivial

What can we conclude about integer solutions?

Corollary: If reduced system has no integer solutions, neither does the original system.

Not true: Reduced system has integer solutions => original system does too.



Key problem: Multiplying one inequality by b_1 and other by d_1 is not guaranteed to preserve "integrality" (cf. equalities)

Exact projection: If all upper bound coefficients b_i or all lower bound coefficients d_i happen to be 1, then integer solution to reduced system implies integer solution to original system.

Theorem: If $(y_1, z_1, ...)$ is an integer vector that satisfies the reduced system in FM elimination, then $(x_1, y_1, z_1 ...)$ satisfies the original system if there exists an integer x_1 between

$\lceil max(1/d_1(c_3 + d_2 y_1 + d_3 z_1 + ...), ...) \rceil$ (over all lower bounds)

and

$\lfloor min(1/b_1(c_2 + b_2 y_1 + b_3 z_1 + ...), ...) \rfloor$ (over all upper bounds).

Proof: trivial

Enumeration: Given a system $Ax \leq b$, we can use Fourier-Motzkin elimination to generate a loop nest to enumerate all integer points that satisfy system as follows:

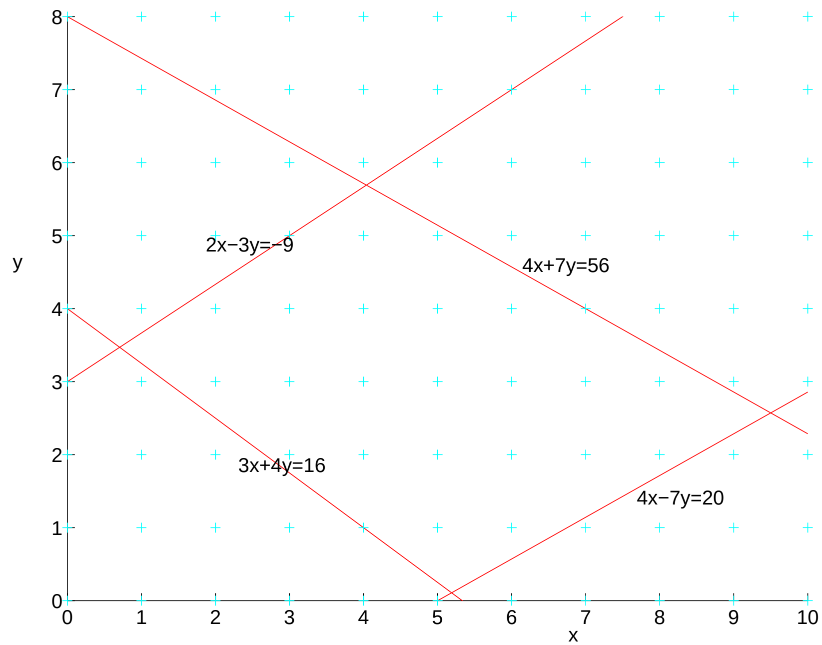
- pick an order to eliminate variables (this will be the order of variables from innermost loop to outermost loop)
- eliminate variables in that order to generate upper and lower bounds for loops as shown in theorem in previous slide

Remark: if polyhedron has no integer points, then the lower bound of some loop in the loop nest will be bigger than the upper bound of that loop

Existence: Given a system $Ax \leq b$, we can use Fourier-Motzkin elimination to project down to a single variable.

- If the reduced system has no integer solutions, then original system has no integer solutions either.
- If the reduced system has integer solutions and all projections were exact, then original system has integer solutions too.
- If reduced system has integer solutions and some projections were no exact, be conservative and assume that original system has integer solutions.

More accurate algorithm for determining existence



Just because there are integers between $4/37$ and $74/13$, we cannot assume there are integers in feasible region.

However, if gap between lower and upper bounds is greater than or equal to 1 for some integer value of y, there must be an integer in feasible region.

Dark shadow: region of y for which gap between upper and lower bounds of x is guaranteed to be greater than or equal to 1.

Determining dark shadow region:

Ordinary FM elimination:

$x \leq u$, $x \geq l => u \geq l$

Dark shadow:

$x \leq u$, $x \geq l => u \geq l + 1$

For our example, dark shadow projection along x gives system

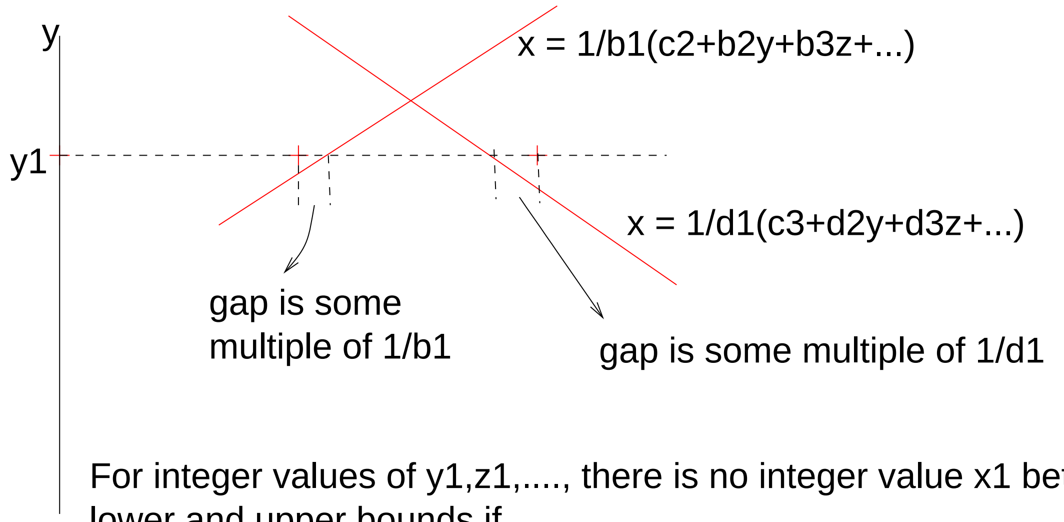
$$
\begin{aligned}
5 + 7y/4 &\geq 16/3 - 4y/3 + 1 (\text{Inequalities} 3, 1) \\
5 + 7y/4 &\geq -9/2 + 3y/2 + 1 (\text{Inequalities} 3, 4) \\
14 - 7y/4 &\geq 16/3 - 4y/3 + 1 (\text{Inequalities} 2, 1) \\
14 - 7y/4 &\geq -9/2 + 3y/2 + 1 (\text{Inequalities} 2, 4)
\end{aligned}
$$

=>

$66/13 \geq$ y $\geq 16/37$

There is an integer value of y in this range => integer in polyhedron.

More accurate estimate of dark shadow



For integer values of y1,z1,...., there is no integer value x1 between lower and upper bounds if

1/d1(c3+d2y1+d3z1+...) - 1/b1(c2+b2y1+b3z1+...) +1/b1+1/d1 <= 1

This means there is an integer between upper and lower bounds if

1/d1(c3+d2y1+d3z1+...) - 1/b1(c2+b2y1+b3z1+...) +1/b1+1/d1 > 1

To convert this to >=, notice that smallest change of lhs value is 1/b1d1.

So the inequality is

1/d1(c3+d2y1+d3z1+...) - 1/b1(c2+b2y1+b3z1+...) +1/b1+1/d1 >= 1 + 1/b1d1

=>

1/d1(c3+d2y1+d3z1+...) - 1/b1(c2+b2y1+b3z1+...) >= (1 - 1/b1)(1 - 1/d1)

Note: If $(b_1 = 1)$ or $(d_1 = 1)$, dark shadow constraint = real shadow constraint

Example:

$$3x \geq 16 - 4y$$
$$4x \leq 20 + 7y$$

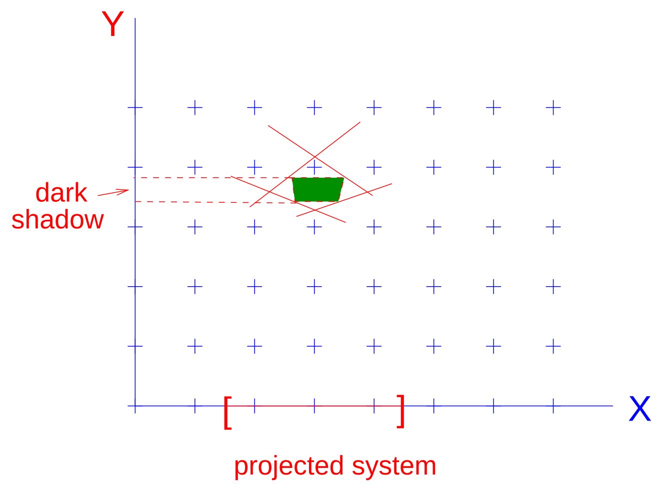
Real shadow: $(20 + 7y) * 3 \geq 4(16 - 4y)$

Dark shadow: $(20 + 7y) * 3 - 4(16 - 4y) \geq 12$

Dark shadow (improved): $(20 + 7y) * 3 - 4(16 - 4y) \geq 6$

What if dark shadow has no integers?

There may still be integer points nestled closely between an upper and lower bound.

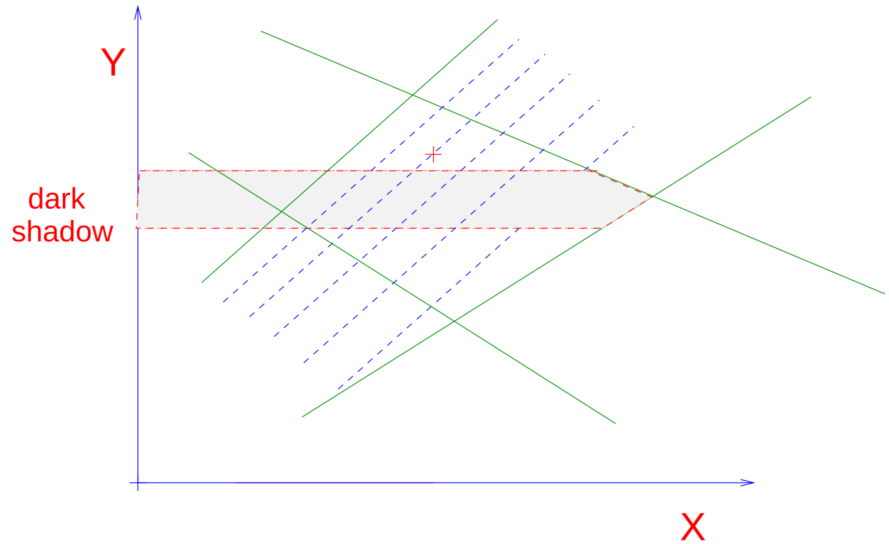


Conservative approach:

- if dark shadow has integer points, deduce correctly that original system has integer solutions
- if dark shadow has no integer points, declare conservatively that original system may have integer solutions

Another alternative: if dark shadow has no integer points, try enumeration

One enumeration idea: splintering



Scan the corners with hyperplanes, looking for integer points.

Generate a succession of problems in which each lower bound is replaced with a sequence of hyperplanes. How many hyperplanes are needed?

Equation for lower bound: x = 1/b1(c2+b2y+b3z+....)

Hyperplanes:

x = 1/b1(c2+b2y+b3z+....)

x = 1/b1(c2+b2y+b3z+....)+ 1/b1

x = 1/b1(c2+b2y+b3z+....)+ 2/b1

x = 1/b1(c2+b2y+b3z+....)+ 3/b1

....

x = 1/b1(c2+b2y+b3z+....)+ 1 (in dark shadow region; if this is integer, so is)

Engineering

- Use matrices and vectors to represent inequalities.

$$\begin{pmatrix} -3 & -4 \\ 4 & 7 \\ 4 & -7 \\ -2 & 3 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} -16 \\ 56 \\ 20 \\ 9 \end{bmatrix}$$

  - lower bounds and upper bounds for a variable can be determined by inspecting signs of entries in column for that variable
  - easy to tell if exact projection is being carried out
  - Fourier-Motzkin elimination is carried out by row operations on pairs of lower and upper bounds. For example, eliminating x:

$$\begin{pmatrix} 0 & 5 \\ 0 & -37 \\ 0 & 13 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 104 \\ -4 \\ 74 \\ 38 \end{bmatrix}$$

- Dark shadow and real shadow computations should be carried out simultaneously to share work (only vector on rhs is different)
- Handle equalities first to reduce number of equations. Find (parameterized) solution to equalities and substitute solution into inequalities.
- Keep co-efficients small by dividing an inequality by gcd of co-efficients if gcd is not 1.
- Check for redundant and contradictory constraints.
- Do exact projections wherever possible.

- Eliminate equations with semi-constrained variables (no upper or no lower bound).

```
DO 10 I = 1, N
   X(I) = ...X(I-1)...
```

```
Flow dependence:
Iw = Ir - 1
1 <= Iw <= Ir <= N
```

`N` only has an lower bound (`N >= Ir`) which can always be satisfied given any values of `(Ir,Iw)`. So eliminate the constraint from consideration.