# Problem Set #2

This problem set is due at the start of class on Thursday, March 8th.

1. Consider the following "marriage game" corresponding to the version of the stable marriage problem in which preferences are strict. The action set of an agent corresponds to the different possible preferences that he or she can submit. Given a preference profile, the agents are matched according to the unique man-optimal matching (e.g., using the deferred acceptance algorithm). Consider an instance $I$ of this game with associated true preference profile $P$. Let $M_{\text{MAN}}$ denote the unique man-optimal matching under $P$. Let $P'$ denote the preference profile that agrees with $P$ for each man, and that replaces the preferences in $P$ of each woman $y$ as follows: (1) if $y$ is unmatched in $M_{\text{MAN}}$, then $y$'s preference list in $P'$ is empty (i.e., $y$ insists on remaining single); (2) if $y$ is matched to a man $x$ in $M_{\text{MAN}}$, then $y$'s preference list in $P'$ consists of $x$ alone (i.e., $y$ claims that she would prefer to marry $x$, and failing that, she insists on remaining single). Prove that $P'$ is a pure Nash equilibrium for instance $I$.

2. In this question we revisit the marriage game of question 1, once again assuming that preferences are strict. Consider an instance of this game with associated true preference profile $P$. Let $M_{\text{WOMAN}}$ denote the unique woman-optimal matching under $P$. Let $P'$ denote the preference profile that agrees with $P$ for each man, and where the preferences of each woman $y$ are determined as follows: (1) if $y$ is unmatched in $M_{\text{WOMAN}}$, then $y$'s preference list in $P'$ is the same as in $P$; (2) if $y$ is matched to a man $x$ in $M_{\text{WOMAN}}$, then $y$'s preference list in $P'$ is the prefix of her preference list in $P$ that terminates with $x$. Let $A$ denote the set of all agents, and let $B$ denote a subset of the women. Let $P''$ denote a preference profile that agrees with $P'$ for every agent in $A - B$. Let $M'$ (resp., $M''$) denote the unique man-optimal matching under $P'$ (resp., $P''$). Prove that if some woman in $B$ prefers $M''$ to $M'$, then some woman in $B$ likes $M'$ at least as well as $M''$. Hint: Make use of the gender-interchanged version of a result presented in class related to groups of men "lying" in the deferred acceptance algorithm. Extra hint (added 2/26/12): It is known that, when preferences are strict, all stable matchings have the same associated set of unmatched men (resp., women); make use of this fact, which you are not required to prove.

3. This question is concerned with the restricted version of the house allocation problem in which agent preferences are strict. For such instances, the Top Trading Cycles (TTC) algorithm produces a unique allocation. Consider an instance in which the true preferences of the agents are given by strict preference profile $P$. Let $A$ denote the set of all agents, and let $B$ denote a subset of $A$. Let $P'$ denote a strict preference profile such that the preferences expressed by each agent in $A - B$ are the same as in $P$. Let $x$ denote the TTC allocation corresponding to $P$, and let $x'$ denote the TTC allocation corresponding to $P'$. Prove that if some agent in $B$ prefers $x'$ to $x$, then some agent in $B$ prefers $x$ to $x'$.

4. We now revisit the previous question, but allowing arbitrary preferences. When ties in preference are allowed, the TTC algorithm can produce more than one possible output. On the other hand, if we adopt a suitable tie-breaking convention, then we can ensure that TTC has a unique output. In this question, assume that all ties in preference are broken in favor of the lower-numbered house. So, for example, if an agent is indifferent between houses 2, 8, and 13, then when we run TTC, we proceed as if this agents prefers house 2 to house 8, and house 8 to house 13. Let us call this modified TTC algorithm TTC*. Consider an instance in which the true preferences of the agents are given by preference profile $P$, which allows ties. Let $A$ denote the set of all agents, and let $B$ denote a subset of $A$. Let $P'$ be another preference profile such that the preferences expressed by any agent in $A - B$ are the same in $P$ as in $P'$. Let $x$ denote the TTC* allocation corresponding to $P$, and let $x'$ denote the TTC* allocation corresponding to $P'$. Prove or disprove: If some agent in $B$ prefers $x'$ to $x$, then some agent in $B$ likes $x$ at least as well as $x'$.