Group Strategyproof Pareto-Stable Marriage with Indifferences via the Generalized Assignment Game*

Nevzat Onur Domaniç, Chi-Kit Lam, and C. Gregory Plaxton

University of Texas at Austin, Austin, TX 78712, USA.
{onur, geocklam, plaxton}@cs.utexas.edu

Abstract. We study the variant of the stable marriage problem in which the preferences of the agents are allowed to include indifferences. We present a mechanism for producing Pareto-stable matchings in stable marriage markets with indifferences that is group strategyproof for one side of the market. Our key technique involves modeling the stable marriage market as a generalized assignment game. We also show that our mechanism can be implemented efficiently. These results can be extended to the college admissions problem with indifferences.

1 Introduction

The stable marriage problem was first introduced by Gale and Shapley [13]. The stable marriage market involves a set of men and women, where each agent has ordinal preferences over the agents of the opposite sex. The goal is to find a disjoint set of man-woman pairs, called a *matching*, such that no other man-woman pair prefers each other to their partners in the matching. Such matchings are said to be *stable*. When preferences are strict, a unique man-optimal stable matching exists and can be computed by the man-proposing deferred acceptance algorithm of Gale and Shapley [13]. A mechanism is said to be *group strategyproof for the men* if no coalition of men can be simultaneously matched to strictly preferred partners by misrepresenting their preferences. Dubins and Freedman [8] show that the mechanism that produces man-optimal matchings is group strategyproof for the men when preferences are strict. In our work, we focus on group strategyproofness for the men, since no stable mechanism is strategyproof for both men and women [19].

We remark that the notion of group strategyproofness used here assumes no side payments within the coalition of men. It is known that group strategyproofness for the men is impossible for the stable marriage problem with strict preferences when side payments are allowed [21, Chap. 4]. This notion of group strategyproofness is also different from strong group strategyproofness, in which at least one man in the coalition gets matched to a strictly preferred partner while the other men in the coalition get matched to weakly preferred

* This research was supported by NSF Grant CCF–1217980.

partners. It is known that strong group strategyproofness for the men is impossible for the stable marriage problem with strict preferences [8, attributed to Gale].

Indifferences in the preferences of agents arise naturally in real-world applications such as school choice [1, 10, 11]. For the marriage problem with indifferences, Sotomayor [24] argues that Pareto-stability is an appropriate solution concept. A matching is said to be *weakly stable* if no man-woman pair strictly prefers each other to their partners in the matching. A matching is said to be *Pareto-optimal* if there is no other matching that is strictly preferred by some agent and weakly preferred by all agents. If a matching is both weakly stable and Pareto-optimal, it is said to be *Pareto-stable*.

Weakly stable matchings, unlike strongly stable or super-stable matchings [14], always exist. However, not all weakly stable matchings are Pareto-optimal [24]. Pareto-stable matchings can be obtained by applying successive Pareto-improvements to weakly stable matchings. Erdil and Ergin [10, 11] show that this procedure can be carried out efficiently. Pareto-stable matchings also exist and can be computed in strongly polynomial time for many-to-many matchings [2] and multi-unit matchings [3]. Instead of using the characterization of Pareto-improvement chains and cycles, Kamiyama [15] gives another efficient algorithm for many-to-many matchings based on rank-maximal matchings. However, none of these mechanisms addresses strategyproofness.

We remark that the notion of Pareto-optimality here is different from *man-Pareto-optimality*, which only takes into account the preferences of the men. It is known that man-Pareto-optimality is not compatible with strategyproofness for the stable marriage problem with indifferences [10, 16]. The notion of Pareto-optimality here is also different from *Pareto-optimality in expected utility*, which permits Pareto-domination by non-pure outcomes. A result of Zhou [25] implies that Pareto-optimality in expected utility is not compatible with strategyproofness for the stable marriage problem with indifferences.

Until recently, it was not known whether a strategyproof Pareto-stable mechanism exists. In our recent workshop paper [7], we present a generalization of the deferred acceptance mechanism that is Pareto-stable and strategyproof for the men. If the market has n agents, our implementation of this mechanism runs in $O(n^4)$ time, matching the time bound of the algorithm of Erdil and Ergin [10, 11][1]. The proof of strategyproofness relies on reasoning about a certain threshold concept in the stable marriage market, and this approach seems difficult to extend to address group strategyproofness.

In this paper, we introduce a new technique useful for investigating incentive compatibility for coalitions of men. We present a Pareto-stable mechanism for the stable marriage problem with indifferences that is provably group strategyproof

[1] The algorithm of Erdil and Ergin proceeds in two phases. In the first phase, ties are broken arbitrarily and the deferred acceptance algorithm is used to obtain a weakly stable matching. In the second phase, a sequence of Pareto-improvements are applied until a Pareto-stable matching is reached. In App. A in the full version of [7], we show that this algorithm does not provide a strategyproof mechanism.

for the men, by modeling the stable marriage market as an appropriate form of the generalized assignment game. In Sect. 4 and the full version [6, App. B and C] of this paper, we show that this mechanism coincides with the generalization of the deferred acceptance mechanism presented in [7]. Thus we obtain an $O(n^4)$-time group strategyproof Pareto-stable mechanism.

The generalized assignment game. The assignment game, introduced by Shapley and Shubik [22], involves a two-sided matching market with monetary transfer in which agents have unit-slope linear utility functions. This model has been generalized to allow agents to have continuous, invertible, and increasing utility functions [4, 5, 18]. Some models that generalize both the assignment game and the stable marriage problems have also been developed, but their models are not concerned with the strategic behavior of agents [12, 23]. The formulation of the generalized assignment game in this paper follows the presentation of Demange and Gale [5].

In their paper, Demange and Gale establish various elegant properties of the generalized assignment game, such as the lattice property and the existence of one-sided optimal outcomes. (One-sided optimality or man-optimality is a stronger notion than one-sided Pareto-optimality or man-Pareto-optimality.) These properties are known to hold for the stable marriage market in the case of strict preferences [17, attributed to Conway], but fail in the case of weak preferences [21, Chap. 2]. Given the similarities between stable marriage markets and generalized assignment games, it is natural to ask whether stable marriage markets can be modeled as generalized assignment games. Demange and Gale discuss this question and state that "the model of [Gale and Shapley] is not a special case of our model". The basic obstacle is that it is unclear how to model an agent's preferences within the framework of a generalized assignment game: on the one hand, even though ordinal preferences can be converted into numeric utility values, such preferences are expressed in a manner that is independent of any monetary transfer; on the other hand, the framework demands that there is an amount of money that makes an agent indifferent between any two agents on the other side of the market.

In Sect. 2, we review key concepts in the work of Demange and Gale, and introduce the *tiered-slope market* as a special form of the generalized assignment game in which the slopes of the utility functions are powers of a large fixed number. Then, in Sect. 3, we describe our approach for converting a stable marriage market with indifferences into an associated tiered-slope market. While these are both two-sided markets that involve the same set of agents, the utilities achieved under an outcome in the associated tiered-slope market may not be equal to the utilities under a corresponding solution in the stable marriage market. Nevertheless, we are able to establish useful relationships between certain sets of solutions to these two markets.

Our first such result, Theorem 2, shows that Pareto-stability in the stable marriage market with indifferences follows from stability in the associated tiered-slope market, even though it does not follow from weak stability in the stable marriage market with indifferences. This can be seen as a partial analogue to the case of strict preferences, in which stability in the stable marriage market implies Pareto-stability [13]. This also demonstrates that, in addition to using the deferred acceptance procedure to solve the generalized assignment game [4], we can use the generalized assignment game to solve the stable marriage problem with indifferences.

In Lemma 5, we establish that the utility achieved by any man in a man-optimal solution to the associated tiered-slope market uniquely determines the tier of preference to which that man is matched in the stable marriage market with indifferences. Another consequence of this lemma is that any matched man in a man-optimal outcome of the associated tiered-slope market receives at least one unit of money from his partner. We can then deduce that if a man strictly prefers his partner to a woman, then the woman has to offer a large amount of money in order for the man to be indifferent between her offer and that of his partner. Since individual rationality prevents any woman from offering such a large amount of money, this explains how we overcome the obstacle of any man being matched with a less preferred woman in exchange for a sufficiently large payment.

A key result established by Demange and Gale is that the man-optimal mechanism is group strategyproof for the men. Using this result and Lemma 5, we are able to show in Theorem 3 that group strategyproofness for the men in the stable marriage market with indifferences is achieved by man-optimality in the associated tiered-slope market, even though it is incompatible with man-Pareto-optimality in the stable marriage market with indifferences [10, 16]. This can be seen as a partial analogue to the case of strict preferences, in which man-optimality implies group strategyproofness [8].

Extending to the college admissions problem. We also consider the settings of incomplete preference lists and one-to-many matchings, in which efficient Pareto-stable mechanisms are known to exist [2, 3, 10, 11, 15]. Preference lists are incomplete when an agent declares another agent of the opposite sex to be unacceptable. Our mechanism is able to support such incomplete preference lists through an appropriate choice of the reserve utilities of the agents in the associated tiered-slope market. In fact, our mechanism also supports indifference between being unmatched and being matched to some partner.

The one-to-many variant of the stable marriage problem with indifferences is the college admissions problem with indifferences. In this model, students and colleges play the roles of men and women, respectively, and colleges are allowed to be matched with multiple students, up to their capacities. We provide the formal definition of the model in the full version [6, App. D] of this paper. By a simple reduction from college admissions markets to stable marriage markets, our mechanism is group strategyproof for the students[2] and produces a Pareto-stable matching in polynomial time.

Organization of this paper. In Sect. 2, we review the generalized assignment game and define the tiered-slope market. In Sect. 3, we introduce the tiered-

[2] A stable mechanism can be strategyproof only for the side having unit demand, namely the students [20].

slope markets associated with the stable marriage markets with indifferences, and use them to obtain a group strategyproof, Pareto-stable mechanism. In Sect. 4 and the full version [6, App. B and C] of this paper, we discuss efficient implementations of the mechanism and its relationship with the generalization of the deferred acceptance algorithm presented in [7]. Due to space limitations, some of the proofs are omitted from this paper. See the full version [6] for all of the proof details.

2 Tiered-Slope Market

The generalized assignment game studied by Demange and Gale [5] involves two disjoint sets I and J of agents, which we call *men* and *women* respectively. We assume that the sets I and J do not contain the element 0, which we use to denote being unmatched. For each man $i \in I$ and woman $j \in J$, the compensation function $f_{i,j}(u_i)$ represents the compensation that man i needs to receive in order to attain utility u_i when he is matched to woman j. Similarly, for each man $i \in I$ and woman $j \in J$, the compensation function $g_{i,j}(v_j)$ represents the compensation that woman j needs to receive in order to attain utility v_j when she is matched to man i. Moreover, each man $i \in I$ has a reserve utility r_i and each woman $j \in J$ has a reserve utility s_j.

In this paper, we assume that the compensation functions are of the form

$$f_{i,j}(u_i) = u_i \lambda^{-a_{i,j}} \qquad \text{and} \qquad g_{i,j}(v_j) = v_j - (b_{i,j}N + \pi_i)$$

and the reserve utilities are of the form

$$r_i = \pi_i \lambda^{a_{i,0}} \qquad \text{and} \qquad s_j = b_{0,j}N,$$

where

$$\pi \in \mathbb{Z}^I; \quad N \in \mathbb{Z}; \quad \lambda \in \mathbb{Z}; \quad a \in \mathbb{Z}^{I \times (J \cup \{0\})}; \quad b \in \mathbb{Z}^{(I \cup \{0\}) \times J}$$

such that $N > \max_{i \in I} \pi_i \geq \min_{i \in I} \pi_i \geq 1$ and

$$\lambda \geq \max_{(i,j) \in (I \cup \{0\}) \times J} (b_{i,j} + 1)N \geq \min_{(i,j) \in (I \cup \{0\}) \times J} (b_{i,j} + 1)N \geq N.$$

We denote this *tiered-slope market* as $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$. When $a_{i,j} = 0$ for every man $i \in I$ and woman $j \in J \cup \{0\}$, this becomes a *unit-slope market* $(I, J, \pi, N, \lambda, 0, b)$. Notice that the compensation functions in a unit-slope market coincide with those in the assignment game [22] where buyer $j \in J$ has a valuation of $b_{i,j}N + \pi_i$ on house $i \in I$. For better readability, we write $\exp_\lambda(\xi)$ to denote λ^ξ.

A *matching* is a function $\mu \colon I \to J \cup \{0\}$ such that for any woman $j \in J$, we have $\mu(i) = j$ for at most one man $i \in I$. Given a matching μ and a woman $j \in J$, we denote

$$\mu(j) = \begin{cases} i & \text{if } \mu(i) = j \\ 0 & \text{if there is no man } i \in I \text{ such that } \mu(i) = j \end{cases}$$

An *outcome* is a triple (μ, u, v), where μ is a matching, $u \in \mathbb{R}^I$ is the utility vector of the men, and $v \in \mathbb{R}^J$ is the utility vector of the women. An outcome (μ, u, v) is *feasible* if the following conditions hold for every man $i \in I$ and woman $j \in J$.

1. If $\mu(i) = j$, then $f_{i,j}(u_i) + g_{i,j}(v_j) \leq 0$.
2. If $\mu(i) = 0$, then $u_i = r_i$.
3. If $\mu(j) = 0$, then $v_j = s_j$.

A feasible outcome (μ, u, v) is *individually rational* if $u_i \geq r_i$ and $v_j \geq s_j$ for every man $i \in I$ and woman $j \in J$. An individually rational outcome (μ, u, v) is *stable* if $f_{i,j}(u_i) + g_{i,j}(v_j) \geq 0$ for every man $i \in I$ and woman $j \in J$.

A stable outcome (μ, u, v) is *man-optimal* if for any stable outcome (μ', u', v') we have $u_i \geq u'_i$ for every man $i \in I$. It has been shown that man-optimal outcomes always exist [5, Property 2]. Theorem 1 below provides a useful group strategyproofness result for man-optimal outcomes.

Theorem 1. *Let (μ, u, v) and (μ', u', v') be man-optimal outcomes of tiered-slope markets $(I, J, \pi, N, \lambda, a, b)$ and $(I, J, \pi, N, \lambda, a', b)$, respectively. If $a \neq a'$, then there exists a man $i_0 \in I$ and a woman $j_0 \in J \cup \{0\}$ with $a_{i_0,j_0} \neq a'_{i_0,j_0}$ such that $u_{i_0} \geq u'_{i_0} \exp_\lambda(a_{i_0,\mu'(i_0)} - a'_{i_0,\mu'(i_0)})$.*

Proof. This follows directly from [5, Theorem 2], which establishes group strategyproofness for the men in the generalized assignment game with no side payments. Notice that the value $u'_{i_0} \exp_\lambda(a_{i_0,\mu'(i_0)} - a'_{i_0,\mu'(i_0)})$ is the true utility of man i_0 under matching μ' as defined in their paper, both in the case of being matched to $\mu'(i_0) \neq 0$ with compensation $u'_{i_0} \exp_\lambda(-a'_{i_0,\mu'(i_0)})$ and in the case of being unmatched. □

3 Stable Marriage with Indifferences

The stable marriage market involves a set I of men and a set J of women. We assume that the sets I and J are disjoint and do not contain the element 0, which we use to denote being unmatched. The preference relation of each man $i \in I$ is specified by a binary relation $\succeq_i$ over $J \cup \{0\}$ that satisfies transitivity and totality. To allow indifferences, the preference relation is not required to satisfy anti-symmetry. Similarly, the preference relation of each woman $j \in J$ is specified by a binary relation $\succeq_j$ over $I \cup \{0\}$ that satisfies transitivity and totality. We denote this *stable marriage market* as $(I, J, (\succeq_i)_{i \in I}, (\succeq_j)_{j \in J})$.

A *matching* is a function $\mu \colon I \to J \cup \{0\}$ such that for any woman $j \in J$, we have $\mu(i) = j$ for at most one man $i \in I$. Given a matching μ and a woman $j \in J$, we denote

$$\mu(j) = \begin{cases} i & \text{if } \mu(i) = j \\ 0 & \text{if there is no man } i \in I \text{ such that } \mu(i) = j \end{cases}$$

A matching μ is *individually rational* if $j \succeq_i 0$ and $i \succeq_j 0$ for every man $i \in I$ and woman $j \in J$ such that $\mu(i) = j$. An individually rational matching μ is *weakly* *stable* if for any man $i \in I$ and woman $j \in J$, either $\mu(i) \succeq_i j$ or $\mu(j) \succeq_j i$. (Otherwise, such a man i and woman j form a *strongly blocking pair*.)

For any matchings μ and μ', we say that the binary relation $\mu \succeq \mu'$ holds if $\mu(i) \succeq_i \mu'(i)$ and $\mu(j) \succeq_j \mu'(j)$ for every man $i \in I$ and woman $j \in J$. A weakly stable matching μ is *Pareto-stable* if for any matching μ' such that $\mu' \succeq \mu$, we have $\mu \succeq \mu'$. (Otherwise, the matching μ is not *Pareto-optimal* because it is *Pareto-dominated* by the matching μ'.)

A *mechanism* is an algorithm that, given a stable marriage market $(I, J, (\succeq_i)_{i \in I}, (\succeq_j)_{j \in J})$, produces a matching μ. A mechanism is said to be *group strategyproof (for the men)* if for any two different preference profiles $(\succeq_i)_{i \in I}$ and $(\succeq'_i)_{i \in I}$, there exists a man $i_0 \in I$ with preference relation $\succeq_{i_0}$ different from $\succeq'_{i_0}$ such that $\mu(i_0) \succeq_{i_0} \mu'(i_0)$, where μ and μ' are the matchings produced by the mechanism given $(I, J, (\succeq_i)_{i \in I}, (\succeq_j)_{j \in J})$ and $(I, J, (\succeq'_i)_{i \in I}, (\succeq_j)_{j \in J})$ respectively. (Such a man i_0 belongs to the coalition but is not matched to a strictly preferred woman by expressing preference relation $\succeq'_{i_0}$ instead of his true preference relation $\succeq_{i_0}$.)

3.1 The Associated Tiered-Slope Market

We construct the *tiered-slope market* $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$ *associated with stable marriage market* $(I, J, (\succeq_i)_{i \in I}, (\succeq_j)_{j \in J})$ as follows. We take $N \geq |I| + 1$ and associate with each man $i \in I$ a fixed and distinct priority $\pi_i \in \{1, 2, \ldots, |I|\}$. We convert the preference relations $(\succeq_i)_{i \in I}$ of the men to integer-valued (non-transferable) utilities $a \in \mathbb{Z}^{I \times (J \cup \{0\})}$ such that for every man $i \in I$ and women $j_1, j_2 \in J \cup \{0\}$, we have $j_1 \succeq_i j_2$ if and only if $a_{i,j_1} \geq a_{i,j_2}$. Similarly, we convert the preference relations $(\succeq_j)_{j \in J}$ of the women to integer-valued (non-transferable) utilities $b \in \mathbb{Z}^{(I \cup \{0\}) \times J}$ such that for every woman $j \in J$ and men $i_1, i_2 \in I \cup \{0\}$, we have $i_1 \succeq_j i_2$ if and only if $b_{i_1,j} \geq b_{i_2,j} \geq 1$. Finally, we take

$$\lambda = \max_{\substack{i \in I \cup \{0\} \\ j \in J}} (b_{i,j} + 1)N.$$

In order to achieve group strategyproofness, we require that N and π should not depend on the preferences $(\succeq_i)_{i \in I}$ of the men. We further require that b does not depend on the preferences $(\succeq_i)_{i \in I}$ of the men, and that a_{i_0,j_0} does not depend on the other preferences $(\succeq_i)_{i \in I \setminus \{i_0\}}$ for any man $i_0 \in I$ and woman $j_0 \in J \cup \{0\}$. In other words, a man $i_0 \in I$ is only able to manipulate his own utilities $(a_{i_0,j})_{j \in J \cup \{0\}}$. One way to satisfy these conditions is by taking a_{i_0,j_0} to be the number of women $j \in J \cup \{0\}$ such that $j_0 \succeq_{i_0} j$ for every man $i_0 \in I$ and woman $j_0 \in J \cup \{0\}$, and taking b_{i_0,j_0} to be the number of men $i \in I \cup \{0\}$ such that $i_0 \succeq_{j_0} i$ for every man $i_0 \in I \cup \{0\}$ and woman $j_0 \in J$. (These conditions are not used until Sect. 3.3, where we prove group strategyproofness.)

Intuitively, each woman has a compensation function with the same form as a buyer in the assignment game [22]. The valuation $b_{i,j}N + \pi_i$ that woman j assigns to man i has a first-order dependence on the preferences over the men and a second-order dependence on the priorities of the men, which are used to break any ties in her preferences. From the perspective of man i, if he highly prefers a woman j, he assigns a large exponent $a_{i,j}$ in the slope associated with woman j, and thus expects only a small amount of compensation.

3.2 Pareto-Stability

In this subsection, we study the Pareto-stability of matchings in the stable marriage market that correspond to stable outcomes in the associated tiered-slope market. We first show that individual rationality in the associated tiered-slope market implies individual rationality in the stable marriage market (Lemmas 1 and 2). Then, we show that stability in the associated tiered-slope market implies weak stability in the stable marriage market (Lemma 3). Finally, we show that stability in the associated tiered-slope market is sufficient for Pareto-stability in the stable marriage market (Lemma 4 and Theorem 2). The proof of Lemma 4 is given in [6, App. A].

Lemma 1. *Let (μ, u, v) be an individually rational outcome in tiered-slope market $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$. Let $i \in I$ be a man and $j \in J$ be a woman. Then*

$$0 < \exp_\lambda(a_{i,0}) \le u_i < \exp_\lambda(a_{i,\mu(i)}+1) \quad \textit{and} \quad 0 \le b_{0,j}N \le v_j < (b_{\mu(j),j}+1)N.$$

Proof. The lower bounds

$$u_i \ge \pi_i \exp_\lambda(a_{i,0}) \ge \exp_\lambda(a_{i,0}) > 0 \qquad \text{and} \qquad v_j \ge b_{0,j}N \ge 0$$

follow directly from individual rationality. If $\mu(i) = 0$, then feasibility implies $u_i = \pi_i \exp_\lambda(a_{i,0}) < \exp_\lambda(a_{i,0}+1)$. If $\mu(j) = 0$, then feasibility implies $v_j = b_{0,j}N < (b_{0,j}+1)N$. It remains to show that the upper bounds hold when $\mu(i) \ne 0$ and $\mu(j) \ne 0$. Without loss of generality, we may assume that $\mu(i) = j$, so feasibility implies

$$u_i \exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i - v_j) \le 0.$$

Since $u_i \ge 0$ and $v_j \ge 0$, we have

$$u_i \exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i) \le 0 \qquad \text{and} \qquad -(b_{i,j}N + \pi_i - v_j) \le 0.$$

Since $\pi_i < N$ and $b_{i,j}N + \pi_i < \lambda$, we have

$$u_i \exp_\lambda(-a_{i,j}) - \lambda < 0 \qquad \text{and} \qquad -(b_{i,j}N + N - v_j) < 0.$$

Thus $u_i < \exp_\lambda(a_{i,\mu(i)}+1)$ and $v_j < (b_{\mu(j),j}+1)N$. □

Lemma 2 (Individual Rationality). *Let (μ, u, v) be an individually rational outcome in the tiered-slope market $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$ associated with stable marriage market $(I, J, (\succeq_i)_{i\in I}, (\succeq_j)_{j\in J})$. Then μ is an individually rational matching in the stable marriage market.*

Proof. Let $i \in I$ be a man and $j \in J$ be a woman. Then, by Lemma 1, we have

$$\exp_\lambda(a_{i,0}) < \exp_\lambda(a_{i,\mu(i)} + 1) \qquad \text{and} \qquad b_{0,j}N < (b_{\mu(j),j} + 1)N.$$

Thus $a_{i,\mu(i)} + 1 > a_{i,0}$ and $b_{\mu(j),j} + 1 > b_{0,j}$, and hence $a_{i,\mu(i)} \geq a_{i,0}$ and $b_{\mu(j),j} \geq b_{0,j}$. We conclude that $\mu(i) \succeq_i 0$ and $\mu(j) \succeq_j 0$. □

Lemma 3 (Stability). *Let (μ, u, v) be a stable outcome in the tiered-slope market $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$ associated with stable marriage market $(I, J, (\succeq_i)_{i\in I}, (\succeq_j)_{j\in J})$. Then μ is a weakly stable matching in the stable marriage market.*

Proof. Since the outcome (μ, u, v) is individually rational in market $\mathcal{M}$, Lemma 2 implies that the matching μ is individually rational in the stable marriage market. It remains to show that there is no strongly blocking pair.

For the sake of contradiction, suppose there exists a man $i \in I$ and a woman $j \in J$ such that neither $\mu(i) \succeq_i j$ nor $\mu(j) \succeq_j i$. Then $a_{i,j} > a_{i,\mu(i)}$ and $b_{i,j} > b_{\mu(j),j}$. Hence $a_{i,j} \geq a_{i,\mu(i)} + 1$ and $b_{i,j} \geq b_{\mu(j),j} + 1$. Since (μ, u, v) is a stable outcome in $\mathcal{M}$, we have

$$\begin{aligned}
0 &\leq u_i \exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i - v_j) \\
&< \exp_\lambda(a_{i,\mu(i)} + 1)\exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i - (b_{\mu(j),j} + 1)N) \\
&\leq 1 - \pi_i,
\end{aligned}$$

where the second inequality follows from Lemma 1. Thus, $\pi_i < 1$, a contradiction. □

Lemma 4. *Let (μ, u, v) be a stable outcome in the tiered-slope market $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$. Let μ' be an arbitrary matching. Then*

$$\sum_{i\in I}\Big(u_i \exp_\lambda(-a_{i,\mu'(i)}) - \pi_i\Big) \geq \sum_{j\in J}\Big(b_{\mu'(j),j}N - v_j\Big).$$

Furthermore, the inequality is tight if and only if the outcome (μ', u, v) is stable.

Theorem 2 (Pareto-stability). *Let (μ, u, v) be a stable outcome in the tiered-slope market $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$ associated with stable marriage market $(I, J, (\succeq_i)_{i\in I}, (\succeq_j)_{j\in J})$. Then μ is a Pareto-stable matching in the stable marriage market.*

Proof. Since the outcome (μ, u, v) is stable in market $\mathcal{M}$, Lemma 3 implies that the matching μ is weakly stable in the stable marriage market. It remains to show that the matching μ is not Pareto-dominated.

Let μ' be a matching of the stable marriage market such that $\mu' \succeq \mu$. Then $\mu'(i) \succeq_i \mu(i)$ and $\mu'(j) \geq_j \mu(j)$ for every man $i \in I$ and woman $j \in J$. Hence $a_{i,\mu'(i)} \geq a_{i,\mu(i)}$ and $b_{\mu'(j),j} \geq b_{\mu(j),j}$ for every man $i \in I$ and woman $j \in J$. Since $a_{i,\mu'(i)} \geq a_{i,\mu(i)}$ for every man $i \in I$, we have

$$\sum_{i\in I}\Big(u_i \exp_\lambda(-a_{i,\mu'(i)}) - \pi_i\Big) \leq \sum_{i\in I}\Big(u_i \exp_\lambda(-a_{i,\mu(i)}) - \pi_i\Big).$$

Applying Lemma 4 to both sides, we get

$$\sum_{j \in J} \Big(b_{\mu'(j),j} N - v_j \Big) \leq \sum_{j \in J} \Big(b_{\mu(j),j} N - v_j \Big).$$

Since $b_{\mu'(j),j} \geq b_{\mu(j),j}$ for every woman $j \in J$, the inequalities are tight. Hence $a_{i,\mu'(i)} = a_{i,\mu(i)}$ and $b_{\mu'(j),j} = b_{\mu(j),j}$ for every man $i \in I$ and woman $j \in J$. Thus $\mu(i) \succeq_i \mu'(i)$ and $\mu'(j) \succeq_j \mu(j)$ for every man $i \in I$ and woman $j \in J$. We conclude that $\mu \succeq \mu'$. □

3.3 Group Strategyproofness

In this subsection, we study the group strategyproofness of matchings in the stable marriage market that correspond to man-optimal outcomes in the associated tiered-slope market. We first show that the utilities of the men in man-optimal outcomes in the associated tiered-slope market reflect the utilities of the men in the stable marriage market (Lemma 5). Then we prove group strategyproofness in the stable marriage market using group strategyproofness in the associated tiered-slope market (Theorem 3).

Lemma 5. *Let (μ, u, v) be a man-optimal outcome in the tiered-slope market $\mathcal{M} = (I, J, \pi, N, \lambda, a, b)$ associated with stable marriage market $(I, J, (\succeq_i)_{i \in I}, (\succeq_j)_{j \in J})$. Then $\exp_\lambda(a_{i,\mu(i)}) \leq u_i < \exp_\lambda(a_{i,\mu(i)} + 1)$ for every man $i \in I$.*

The proof of Lemma 5 is given in [6, App. A]. Since the compensation received by a man $i \in I$ matched with a woman $\mu(i) \neq 0$ is given by $u_i \exp_\lambda(-a_{i,\mu(i)})$, Lemma 5 implies that the amount of compensation in man-optimal outcomes is at least 1 and at most λ. In fact, no woman is willing to pay more than λ under any individual rational outcome.

Theorem 3 (Group strategyproofness). *If a mechanism produces matchings that correspond to man-optimal outcomes of the tiered-slope markets associated with the stable marriage markets, then it is group strategyproof and Pareto-stable.*

Proof. We have shown Pareto-stability in the stable marriage market in Theorem 2. It remains only to show group strategyproofness.

Let $(I, J, (\succeq_i)_{i \in I}, (\succeq_j)_{j \in J})$ and $(I, J, (\succeq'_i)_{i \in I}, (\succeq_j)_{j \in J})$ be stable marriage markets where $(\succeq_i)_{i \in I}$ and $(\succeq'_i)_{i \in I}$ are different preference profiles. Let $(I, J, \pi, N, \lambda, a, b)$ and $(I, J, \pi, N, \lambda, a', b)$ be the tiered-slope markets associated with stable marriage markets $(I, J, (\succeq_i)_{i \in I}, (\succeq_j)_{j \in J})$ and $(I, J, (\succeq'_i)_{i \in I}, (\succeq_j)_{j \in J})$, respectively. Let (μ, u, v) and (μ', u', v') be man-optimal outcomes of the tiered-slope markets $(I, J, \pi, N, \lambda, a, b)$ and $(I, J, \pi, N, \lambda, a', b)$, respectively.

Since the preference profiles $(\succeq_i)_{i \in I}$ and $(\succeq'_i)_{i \in I}$ are different, we have $a \neq a'$. So, by Theorem 1, there exists a man $i_0 \in I$ and a woman $j_0 \in J \cup \{0\}$ with

$a_{i_0,j_0} \neq a'_{i_0,j_0}$ such that $u_{i_0} \geq u'_{i_0} \exp_\lambda(a_{i_0,\mu'(i_0)} - a'_{i_0,\mu'(i_0)})$. Hence

$$\begin{aligned}\exp_\lambda(a_{i_0,\mu(i_0)} + 1) &> u_{i_0} \\ &\geq \frac{u'_{i_0}}{\exp_\lambda(a'_{i_0,\mu'(i_0)})} \exp_\lambda(a_{i_0,\mu'(i_0)}) \\ &\geq \exp_\lambda(a_{i_0,\mu'(i_0)}),\end{aligned}$$

where the first and third inequalities follow from Lemma 5. This shows that $a_{i_0,\mu(i_0)}+1 > a_{i_0,\mu'(i_0)}$. Hence $a_{i_0,\mu(i_0)} \geq a_{i_0,\mu'(i_0)}$, and we conclude that $\mu(i_0) \succeq_{i_0} \mu'(i_0)$. Also, since $a_{i_0,j_0} \neq a'_{i_0,j_0}$, the preference relations $\succeq_{i_0}$ and $\succeq'_{i_0}$ are different. Therefore, the mechanism is group strategyproof. □

4 Efficient Implementation

The implementation of our group strategyproof Pareto-stable mechanism for stable marriage with indifferences amounts to computing a man-optimal outcome for the associated tiered-slope market. Since all utility functions in the tiered-slope market are linear functions, we can perform this computation using the algorithm of Dütting et al. [9], which was developed for multi-item auctions. If we model each woman j as a non-dummy item in the multi-item auction with price given by utility v_j, then the utility function of each man on each non-dummy item is a linear function of the price with a negative slope. Using the algorithm of Dütting et al., we can compute a man-optimal (envy-free) outcome using $O(n^5)$ arithmetic operations, where n is the total number of agents. Since poly(n) precision is sufficient, our mechanism admits a polynomial-time implementation.

For the purpose of solving the stable marriage problem, it is actually sufficient for a mechanism to produce the matching without the utility vectors u and v of the associated tiered-slope market. In [6, App. B and C], we show that the generalization of the deferred acceptance algorithm presented in [7] can be used to compute a matching that corresponds to a man-optimal outcome for the associated tiered-slope markets. The proof of Theorem 4 is given in [6, App. C.3].

Theorem 4. *There exists an $O(n^4)$-time algorithm that corresponds to a group strategyproof Pareto-stable mechanism for the stable marriage market with indifferences, where n is the total number of men and women.*

References

1. Abdulkadiroğlu, A., Pathak, P.A., Roth, A.E.: Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match. The American Economic Review 99, 1954–1978 (2009)
2. Chen, N.: On computing Pareto stable assignments. In: Proceedings of the 29th International Symposium on Theoretical Aspects of Computer Science. pp. 384–395 (2012)

3. Chen, N., Ghosh, A.: Algorithms for Pareto stable assignment. In: Proceedings of the Third International Workshop on Computational Social Choice. pp. 343–354 (2010)
4. Crawford, V.P., Knoer, E.M.: Job matching with heterogeneous firms and workers. Econometrica 49, 437–450 (1981)
5. Demange, G., Gale, D.: The strategy structure of two-sided matching markets. Econometrica 53, 873–888 (1985)
6. Domaniç, N.O., Lam, C.K., Plaxton, C.G.: Group strategyproof Pareto-stable marriage with indifferences via the generalized assignment game (July 2017), https://arxiv.org/abs/1707.01496
7. Domaniç, N.O., Lam, C.K., Plaxton, C.G.: Strategyproof Pareto-stable mechanisms for two-sided matching with indifferences. In: Fourth International Workshop on Matching under Preferences (April 2017), full version available at https://arxiv.org/abs/1703.10598
8. Dubins, L.E., Freedman, D.A.: Machiavelli and the Gale-Shapley algorithm. American Mathematical Monthly 88, 485–494 (1981)
9. Dütting, P., Henzinger, M., Weber, I.: An expressive mechanism for auctions on the web. ACM Transactions on Economics and Computation 4, 1:1–1:34 (2015)
10. Erdil, A., Ergin, H.: What's the matter with tie-breaking? Improving efficiency in school choice. American Economic Review 98, 669–689 (2008)
11. Erdil, A., Ergin, H.: Two-sided matching with indifferences (2015), working paper
12. Eriksson, K., Karlander, J.: Stable matching in a common generalization of the marriage and assignment models. Discrete Mathematics 217, 135–156 (2000)
13. Gale, D., Shapley, L.S.: College admissions and the stability of marriage. American Mathematical Monthly 69, 9–15 (1962)
14. Irving, R.W.: Stable marriage and indifference. Discrete Applied Mathematics 48, 261–272 (1994)
15. Kamiyama, N.: A new approach to the Pareto stable matching problem. Mathematics of Operations Research 39, 851–862 (2014)
16. Kesten, O.: School choice with consent. The Quarterly Journal of Economics 125, 1297–1348 (2010)
17. Knuth, D.: Marriages Stables. Montreal University Press, Montreal (1976)
18. Quinzii, M.: Core and competitive equilibria with indivisibilities. International Journal of Game Theory 13, 41–60 (1984)
19. Roth, A.E.: The economics of matching: Stability and incentives. Mathematics of Operations Research 7, 617–628 (1982)
20. Roth, A.E.: The college admissions problem is not equivalent to the marriage problem. Journal of Economic Theory 36, 277–288 (1985)
21. Roth, A.E., Sotomayor, M.: Two-sided matching: A study in game-theoretic modeling and analysis. Cambridge University Press, New York (1990)
22. Shapley, L.S., Shubik, M.: The assignment game I: The core. International Journal of Game Theory 1, 111–130 (1971)
23. Sotomayor, M.: Existence of stable outcomes and the lattice property for a unified matching market. Mathematical Social Sciences 39, 119–132 (2000)
24. Sotomayor, M.: The Pareto-stability concept is a natural solution concept for discrete matching markets with indifferences. International Journal of Game Theory 40, 631–644 (2011)
25. Zhou, L.: On a conjecture by Gale about one-sided matching problems. Journal of Economic Theory 52, 123–135 (1990)