CS311H

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Good Morning, Colleagues

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Are there any questions?

Logistics

- Midterm on graph theory, counting, recurrences on Thursday
  - Like last time: hand-written notes allowed. No book or electronic devices.
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- Wed. before Thanksgiving?

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  – v′s neighbors have at most k-2 distinct colors.
  – v can be give a new color, which means X(G) <= k-1, contradiction.

Counting

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- How many ways to choose a dozen donuts if there are 4 types?

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- How many ways are there to place 35 students into 7 groups of 5?

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  - Functions with only one element in range subtracted twice
  - $3^6 - (\binom{3}{1}2^6 - \binom{3}{2})$ = 540

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- Left equation: pick a leader first from n, then there are 2^{n-1} possible subsets of other people.
- Right equation: consider how many committees of size k there are from $k = 1$ to n. For each of these, there are k possible leaders.

Soccer Balls

- The surface of a soccer ball is a planar graph satisfying the following two conditions: (1) Each vertex has degree 3. (2) There are only two kinds of regions: pentagons and hexagons. There are 20 hexagons. Figure out the number of pentagons.

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- Suppose that a planar graph with E edges and V vertices contains no simple circuits of length 4 or less.
  Show that | E | $\leq \frac{5}{3}|V| - \frac{10}{3}$ if $|V| \geq 4$

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Thus we have $x = 12$. So we have 12 pentagons.

Counting subgraphs

- Prove that a fully connected graph K_n has $2^n - 1$ fully connected subgraphs.