Good Morning, Colleagues
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Are there any questions?
Logistics

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  – But we’ve shown some fully formal examples
  – On HW and test you need to be fully formal
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- Bring in chairs?
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• Second homework **due at start of class**
Some questions

- How do you do proof by exhaustive cases?
Some questions

• How do you do proof by exhaustive cases?
  – First establish all the possible solutions
Some questions

• How do you do proof by exhaustive cases?
  – First establish all the possible solutions
  – Then examine them one by one
Prove that...

• Every odd integer is the difference of two squares using direct proof.
Prove that... 

- Every odd integer is the difference of two squares using **direct proof**.
  - Is the converse true? That is, is the difference of two squares always an odd integer?
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• Every odd integer is the difference of two squares using **direct proof**.
  
  – Is the converse true? That is, is the difference of two squares always an odd integer?
Prove by Contradiction

- There are no positive integer solutions to equation

\[ x^2 - y^2 = 1 \]
Uniqueness of Quotients and Remainders

**Thm:** Let $a$ be an integer, $d$ positive integer. Then there are unique integers $q, r$ with $0 \leq r < d$ such that $a = dq + r$
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Peter Stone
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6. Combining 4 and 5, $d(q - q') > r' - r$—Contradicts 3.
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7. Therefore \( q = q' \) (step 2)
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8. Therefore \( r = r' \). QED.
Prove:

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  3. Each of these consists of 50 terms, and each term in \(B\) is larger than the corresponding term in \(A\).
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  3. Each of these consists of 50 terms, and each term in \(B\) is larger than the corresponding term in \(A.\)
  4. Therefore \(A < B \Rightarrow A^2 < AB.\)
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  6. So, \(A^2 < 1/100 \Rightarrow A < 1/10\).
Show that there is no rational number $r$ for which $r^3 + r + 1 = 0$
Prove by direct proof:

• If $a$ and $b$ are real numbers, then $a^2 + b^2 \geq 2ab$. 
Prove by direct proof:

- If $a$ and $b$ are real numbers, then $a^2 + b^2 \geq 2ab$.

- (Tricky problem) The number $100...01$ (with $3n - 1$ zeros where $n$ is a positive integer) is not a prime. (Hint: using identity $x^3 + 1 = (x + 1)(x^2 - x + 1)$.)
Assignments for Tuesday

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- Modules 7,8 with associated readings
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