CS311H

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Recurrences

- In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

Good Morning, Colleagues

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Are there any questions?

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  - Tuesday and Wednesday devoted to review.

Quiz

- If the class has 60 students, how many ways can I divide the class in half?

- If I have 100 Snickers bars, in how many ways can I divide them up among the students in that class?

- In the same class, if 10 of the students are named "Will," 15 are named "William," 5 are named "Bill" and the rest have unique names, how many ways can I write the class members' (first) names in order?

- According to Pascal's identity, what is $\binom{35}{12} + \binom{35}{13}$?

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  Solving for initial conditions, the final recurrence is:
  $T_n = -(3/6)(-1)^n + (7/6) + (1/3)(-2)^n$

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- $\frac{2^{n+1}}{3} + \frac{(-1)^n}{3}$

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  We can now find P_{30} under initial condition $P_0 = 10{,}000$.
  $P_{30} = (1.05)^{30}10{,}000 = \$43{,}219.42$

More Difficult

- Let $ABCDEFGH$ be a regular octagon of side length 1, and O be the center of the octagon. In addition to the sides of the octagon, line segments are drawn from O to each vertex, making a total of 16 line segments. Let a_n be the number of paths (not necessarily simple) of length n along these line segments that start at O and terminate at O. Give a closed form solution of a_n.

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Solution

Let b_n be the number of path of length n that start at O and terminate at A (b_n also works for $BCDEFGH$). Since for the first step, we need move to one of the 8 vertices. Then we get the recurrence relationship that

$$a_n = 8b_{n-1}$$

For b_n, consider the last step, it can be from its two adjacent vertices or from the center O. Thus we have

$$b_n = 2b_{n-1} + a_{n-1}$$

Substituting b_n by $a_{n+1}/8$ we get

$$a_{n+1} - 2a_n - 8a_{n-1} = 0$$

For initial condition, we have $a_0 = 1$ and $a_1 = 0$. The characteristic polynomial is

$$x^2 - 2x - 8$$

which has roots of $x = 4$ and $x = -2$. Thus the solution of the homogeneous recurrence relationship is in form

$$a_n = \alpha(4)^n + \beta(-2)^n$$

Using initial condition, we have

$$\begin{aligned} a_0 &= 1 = \alpha + \beta \\ a_1 &= 0 = (4)\alpha + (-2)\beta \end{aligned}$$

Thus we have and $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$ and the close form solution is

$$a_n = \frac{1}{3}4^n + \frac{2}{3}(-2)^n$$