Recurrences

- In how many ways can a $2 \times n$ rectangular checkerboard be tiled using $1 \times 2$ and $2 \times 2$ pieces?
Good Morning, Colleagues
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Are there any questions?
Logistics

- Midterm on graph theory, counting, recurrences next Thursday
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  - Like last time: hand-written notes allowed. No book or electronic devices.
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  - Like last time: hand-written notes allowed. No book or electronic devices.
  - Tuesday and Wednesday devoted to review.
Quiz

• If the class has 60 students, how many ways can I divide the class in half?

• If I have 100 Snickers bars, in how many ways can I divide them up among the students in that class?

• In the same class, if 10 of the students are named “Will,” 15 are named “William,” 5 are named “Bill” and the rest have unique names, how many ways can I write the class members’ (first) names in order?

• According to Pascal’s identity, what is \( \binom{35}{12} + \binom{35}{13} \)?
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- Find closed form solution for
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● Find closed form solution for
  \[ a_0 = 5, \ a_1 = 2, \text{ and } a_n = -10a_{n-1} - 25a_{n-2}. \]
A Generalization

• Find closed form solution for

\[ T_0 = 1, \ T_1 = 1, \ T_2 = 2, \text{ and } T_n = -2T_{n-1} + T_{n-2} + 2T_{n-3} \]
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The characteristic polynomial is:
\[ r^3 + 2r^2 - r - 2 = (r + 1)(r - 1)(r + 2) \]
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Solution form is:
\[ T_n = \alpha(-1)^n + \beta(1)^n + \gamma(-2)^n = \alpha(-1)^n + \beta + \gamma(-2)^n \]
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So the roots are \( r_1 = -1, r_2 = 1, r_3 = -2 \)
Solution form is:
\[ T_n = a(-1)^n + b(1)^n + c(-2)^n = a(-1)^n + b + c(-2)^n \]
Solving for initial conditions, the final recurrence is:
\[ T_n = -(3/6)(-1)^n + (7/6) + (1/3)(-2)^n \]
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• $\frac{2^{n+1}}{3} + \frac{(-1)^n}{3}$
An Application

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We can now find $P_{30}$ under initial condition $P_0 = 10,000$. 
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We can now find $P_{30}$ under initial condition $P_0 = 10,000$.

$$P_{30} = (1.05)^{30}10,000 = 43,219.42$$
More Difficult

• Let $ABCDEFGH$ be a regular octagon of side length 1, and $O$ be the center of the octagon. In addition to the sides of the octagon, line segments are drawn from $O$ to each vertex, making a total of 16 line segments. Let $a_n$ be the number of paths (not necessarily necessarily simple) of length $n$ along these line segments that start at $O$ and terminate at $O$. Give a closed form solution of $a_n$. 
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Solution

Let $b_n$ be the number of paths of length $n$ that start at $O$ and terminate at $A$ ($b_n$ also works for $BCDEFGH$). Since for the first step, we need move to one of the 8 vertices. Then we get the recurrence relationship that

$$a_n = 8b_{n-1}$$

For $b_n$, consider the last step, it can be from its two adjacent vertices or from the center $O$. Thus we have

$$b_n = 2b_{n-1} + a_{n-1}$$
Substituting $b_n$ by $a_{n+1}/8$ we get

$$a_{n+1} - 2a_n - 8a_{n-1} = 0$$

For initial condition, we have $a_0 = 1$ and $a_1 = 0$. The characteristic polynomial is

$$x^2 - 2x - 8$$

which has roots of $x = 4$ and $x = -2$. Thus the solution of the homogeneous recurrence relationship is in form

$$a_n = \alpha(4)^n + \beta(-2)^n$$
Using initial condition, we have

\[
\begin{align*}
a_0 &= 1 = \alpha + \beta \\
a_1 &= 0 = (4)\alpha + (-2)\beta
\end{align*}
\]

Thus we have and \( \alpha = \frac{1}{3} \) and \( \beta = \frac{2}{3} \) and the close form solution is

\[
a_n = \frac{1}{3}4^n + \frac{2}{3}(-2)^n
\]