Good Morning, Colleagues
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Are there any questions?
Logistics

- Need to postpone office hours Thursday
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  - Available in the early afternoon
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  - Available in the early afternoon
- Modules for next week coming late
Some questions

• Couldn’t come up with proof on own
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  - That’s OK, as long as you understand it
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- Why do bipartite graphs not need to be all connected, but trees do?
For $G = (\{a, b, c, d, e\}, \{(a, b), (e, d), (a, c), (b, c), (e, c), (d, c)\})$

1. Identify all simple paths from $a$ to $e$. 
Definitions

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1. Identify all simple paths from $a$ to $e$.

2. Identify all simple circuits starting and ending at $a$.

3. Identify all cycles starting and ending at $a$. 

Peter Stone
Definitions

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1. Identify all simple paths from $a$ to $e$.
   - $(a, c, e), (a, b, c, e), (a, c, d, e), (a, b, c, d, e)$

2. Identify all simple circuits starting and ending at $a$.
   - $(a, b, c, a), (a, c, b, a), (a, b, c, d, e, c, a), (a, b, c, e, d, c, a),
     (a, c, e, d, c, b, a), (a, c, d, e, c, b, a)$

3. Identify all cycles starting and ending at $a$.
   Subset of the simple circuits: $(a, b, c, a), (a, c, b, a)$
Prove

• For a graph G, if MAX-DEGREE(G) = 3, then any simple circuit is actually a cycle.
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Proof by contradiction:
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1. Assume the simple circuit \((s, \ldots, s)\) is not a cycle.
2. Then there must be a repeated vertex \(v\) so the circuit is \((s, \ldots, v, \ldots, v, \ldots, s)\)
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4. Therefore the vertices preceding and following $v$ in each case must be distinct.
5. So the circuit is $(s, \ldots, a, v, b, \ldots, x, v, y, \ldots, s)$ ($a$ or $y$ could equal $s$, but not both)
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6. Then the degree of \(v\) is at least 4.
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Proof by contradiction:
1. Assume the simple circuit $(s, ..., s)$ is not a cycle.
2. Then there must be a repeated vertex $v$ so the circuit is $(s, ..., v, ..., v, ..., s)$
3. Since the circuit is "simple", no repeated edges.
4. Therefore the vertices preceding and following $v$ in each case must be distinct.
5. So the circuit is $(s, ..., a, v, b, ..., x, v, y, ..., s)$ ($a$ or $y$ could equal $s$, but not both)
6. Then the degree of $v$ is at least 4.
7. $4 > 3 = \text{MAX-DEGREE}(G)$ is a contradiction.
Find a Counterexample

Suppose all vertices of a graph $G$ have been colored. Now suppose that all cycles are found, and it turns out that for each cycle $(v_1, v_2, ..., v_n, v_1)$ that $v_1, ..., v_n$ all have distinct colors. In this case, the coloring must be valid.
Find a Counterexample

Suppose all vertices of a graph $G$ have been colored. Now suppose that all cycles are found, and it turns out that for each cycle $(v_1, v_2, ..., v_n, v_1)$ that $v_1, ..., v_n$ all have distinct colors. In this case, the coloring must be valid.

Create a counterexample using a vertex that doesn’t appear in ANY cycles. Take the graph $G = (\{a, b, c, d\}, \{(a, b), (b, c), (c, a), (a, d)\})$. Then the cycles are $(a, b, c, a), (b, c, a, b), (c, a, b, c)$, none of which contain $d$, so assign the colors: $a$:RED, $b$:BLUE, $c$:GREEN, $d$:RED. Colors are distinct within each cycle, but the color of $d$ clashes with $a$. 
1. Prove that a graph with exactly two vertices with odd degree must contain a path between these two vertices.
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2. Prove that every graph with vertices that each have degree at least 2 contains a cycle.
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Bipartite Graphs, Trees

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2. In a 2-colored Tree with \( n \) vertices, what is the maximum number of vertices that can be one color?
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2. In a 2-colored Tree with $n$ vertices, what is the maximum number of vertices that can be one color?

3. Prove that adding an edge to any Tree will create a cycle.
Bipartite Graphs, Trees

1. Prove all trees are bipartite graphs.
   - \( G \) is a Tree
   - \( \Leftrightarrow \) \( G \) is connected \( \land \) \( G \) has no cycles
   - \( \Rightarrow \) \( G \) has no odd length cycles
   - \( \Leftrightarrow \) \( G \) is bipartite

2. In a 2-colored Tree with \( n \) vertices, what is the maximum number of vertices that can be one color?
   - \( n - 1 \) vertices can have the same color in a star graph, which is a Tree.

3. Prove that adding an edge to any Tree will create a cycle.
Bipartite Graphs

If $G$ is a bipartite graph and the bipartition of $G$ is $X$ and $Y$, then

$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$$
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Ans:

Proof by induction on number of edges:

$P(n) = \text{If } G \text{ is a bipartite graph with } n \text{ edges and the bipartition of } G \text{ is } X \text{ and } Y, \text{ then } \sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$

Base Case: $n = 1$. No. of edges between $X$ and $Y$ is 1.

$$\sum_{v \in X} \deg(v) = 1 = \sum_{v \in Y} \deg(v).$$

Inductive Case: Assume $P(n)$ is true. Remove one edge $e$ between $X$ and $Y$. The resulting graph has $n$ edges, so we can apply the inductive hypothesis. Putting $e$ back adds exactly 1 to both $\sum_{v \in X} \deg(v)$ and $\sum_{v \in Y} \deg(v)$, so we have $P(n+1) = \text{true}$. Hence proved.
Prove that if $G$ with $n$ ($n > 1$) vertices is connected and has $n - 1$ edges, then $G$ is a Tree.
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Proof by induction on the number of vertices
Assignments for Thursday

- Module 14