CS313H
Logic, Sets, and Functions: Honors
Fall 2012

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Are there any questions?

Logistics

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- Modules for next week coming late

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  - That's OK, as long as you understand it
  - Now that you've seen it, you should be able to regenerate it
- Why do bipartite graphs not need to be all connected, but trees do?

Definitions

For $G = (\{a, b, c, d, e\}, \{(a, b), (e, d), (a, c), (b, c), (e, c), (d, c)\})$

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   (a,b,c,a), (a,c,b,a), (a,b,c,d,e,c,a), (a,b,c,e,d,c,a), (a,c,e,d,c,b,a), (a,c,d,e,c,b,a)

3. Identify all cycles starting and ending at a.
   Subset of the simple circuits: (a,b,c,a), (a,c,b,a)

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  5. So the circuit is $(s, ..., a, v, b, ..., x, v, y, ..., s)$ (a or y could equal s, but not both)

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  5. So the circuit is $(s, ..., a, v, b, ..., x, v, y, ..., s)$ (a or y could equal s, but not both)
  6. Then the degree of v is at least 4.
  7. $4 > 3 =$MAX-DEGREE(G) is a contradiction.

Find a Counterexample

Suppose all vertices of a graph G have been colored. Now suppose that all cycles are found, and it turns out that for each cycle $(v_1, v_2, ..., v_n, v_1)$ that $v_1, ..., v_n$ all have distinct colors. In this case, the coloring must be valid.

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Create a counterexample using a vertex that doesn't appear in ANY cycles. Take the graph
$G = (\{a, b, c, d\}, \{(a, b), (b, c), (c, a), (a, d)\})$.
Then the cycles are $(a, b, c, a), (b, c, a, b), (c, a, b, c)$, none of which contain d, so assign the colors: a:RED, b:BLUE, c:GREEN, d:RED. Colors are distinct within each cycle, but the color of d clashes with a.

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1. Prove that a graph with exactly two vertices with odd degree must contain a path between these two vertices.

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2. Prove that every graph with vertices that each have degree at least 2 contains a cycle.

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Bipartite Graphs, Trees

1. Prove all trees are bipartite graphs.
   G is a Tree
   $\Leftrightarrow$ G is connected $\wedge$ G has no cycles
   $\Rightarrow$ G has no odd length cycles
   $\Leftrightarrow$ G is bipartite

2. In a 2-colored Tree with n vertices, what is the maximum number of vertices that can be one color?
   $n - 1$ vertices can have the same color in a star graph, which is a Tree.

3. Prove that adding an edge to any Tree will create a cycle.

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Ans:
Proof by induction on number of edges:
P(n) = If G is a bipartite graph with n edges and the bipartition of G is X and Y, then $\sum_{v \in X} deg(v) = \sum_{v \in Y} deg(v)$
Base Case: n = 1. No. of edges between X and Y is 1. $\sum_{v \in X} deg(v) = 1 = \sum_{v \in Y} deg(v)$.
Inductive Case: Assume P(n) is true. Remove one edge e between X and Y. The resulting graph has n edges, so we can apply the inductive hypothesis. Putting e back adds exactly 1 to both $\sum_{v \in X} deg(v)$ and $\sum_{v \in Y} deg(v)$, so we have P(n+1) = true. Hence proved.

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Proof by induction on the number of vertices

Assignments for Thursday

- Module 14