

CS 343H: Honors Artificial Intelligence

Hidden Markov Models



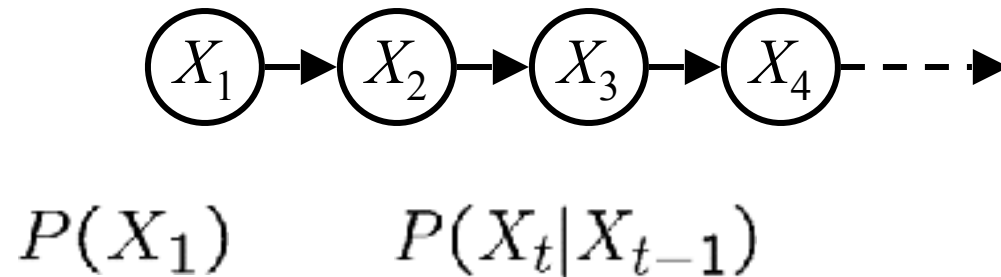
Prof. Peter Stone — The University of Texas at Austin

Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
 - Speech recognition
 - Robot localization
 - User attention
 - Medical monitoring
- Need to introduce time (or space) into our models

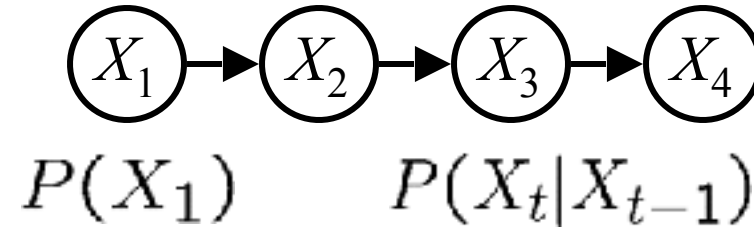
Markov Models

- Value of X at a given time is called the **state**



- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

Joint Distribution of a Markov Model



- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

- More generally:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

Implied Conditional Independencies



- We assumed: $X_3 \perp\!\!\!\perp X_1 \mid X_2$ and $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

- Do we also have $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$?

- Yes! D-Separation

- Or, Proof:

$$\begin{aligned} P(X_1 \mid X_2, X_3, X_4) &= \frac{P(X_1, X_2, X_3, X_4)}{P(X_2, X_3, X_4)} \\ &= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}{\sum_{x_1} P(x_1)P(X_2 \mid x_1)P(X_3 \mid X_2)P(X_4 \mid X_3)} \\ &= \frac{P(X_1, X_2)}{P(X_2)} \\ &= P(X_1 \mid X_2) \end{aligned}$$

Markov Models Recap

- Explicit assumption for all t : $X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Implied conditional independencies:
 - Past variables independent of future variables given the present
i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption: $P(X_t \mid X_{t-1})$ is the same for all t

Conditional Independence

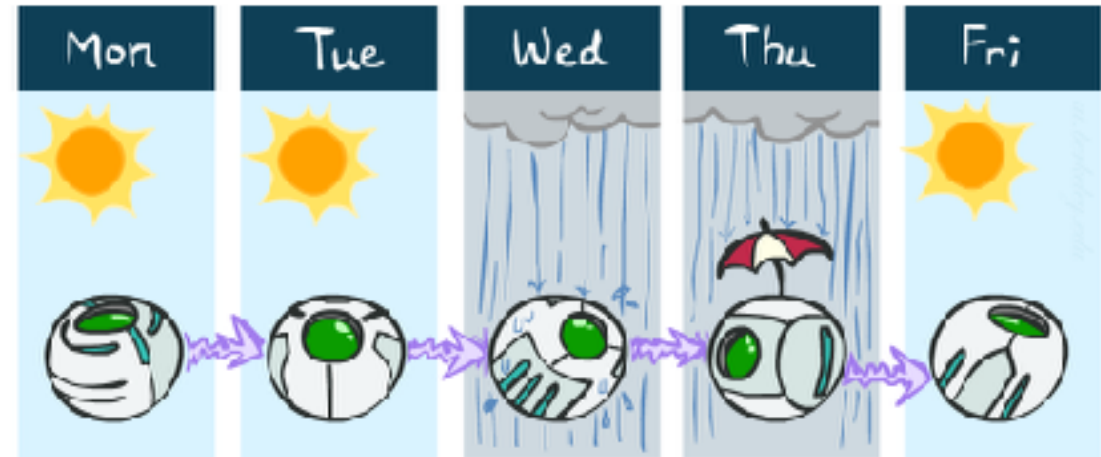


- Basic conditional independence:
 - Past and future independent given the present
 - Each time step only depends on the previous
 - This is the (first order) Markov property (remember MDPs?)
- Note that the chain is just a (growable) BN
 - We can always use generic BN reasoning on it if we truncate the chain at a fixed length

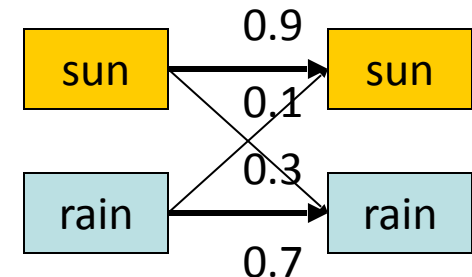
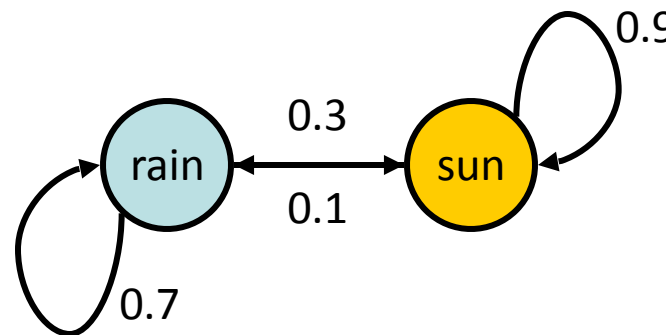
Example Markov Chain: Weather

- States: $X = \{\text{rain}, \text{sun}\}$
- Initial distribution: 1.0 sun
- CPT $P(X_t | X_{t-1})$:

X_{t-1}	X_t	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

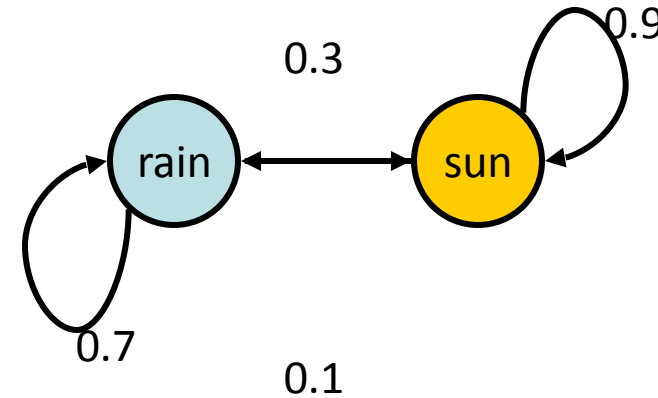


Two new ways of representing the same CPT



Example Markov Chain: Weather

- Initial distribution: 1.0 sun



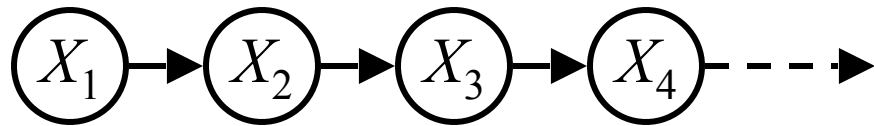
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

Mini-Forward Algorithm

- Question: What's $P(X)$ on some day t ?

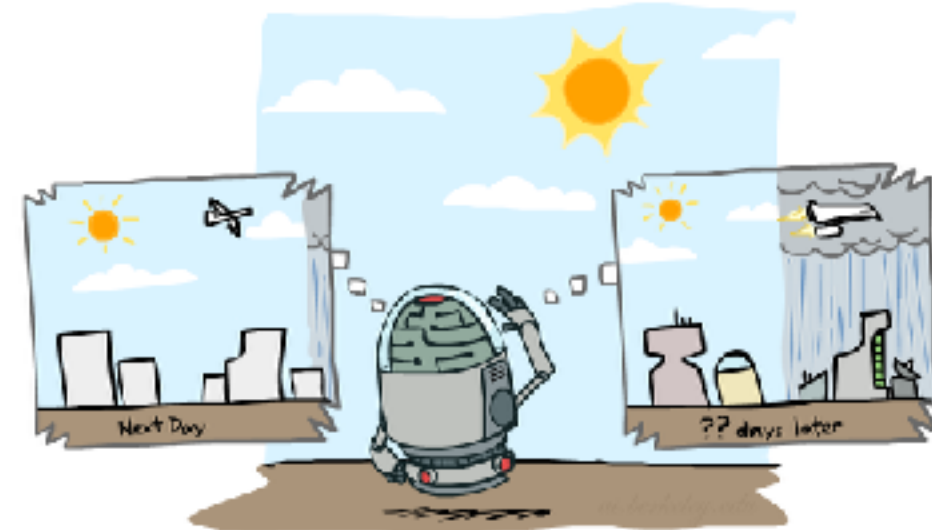


$P(x_1) = \text{known}$

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \end{aligned}$$

← Recursion

← Forward simulation



Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{array}{ccccccc}
 \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty)
 \end{array}$$

- From initial observation of rain

$$\begin{array}{ccccccc}
 \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty)
 \end{array}$$

- From yet another initial distribution $P(X_1)$:

$$\begin{array}{ccc}
 \left\langle \begin{array}{c} p \\ 1-p \end{array} \right\rangle & \dots & \longrightarrow \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & & P(X_\infty)
 \end{array}$$

Stationary Distributions

- For most chains:

- Influence of the initial distribution gets less and less over time.
- The distribution we end up in is independent of the initial distribution

- Stationary distribution:

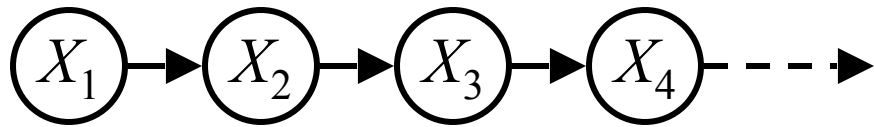
- The distribution we end up with is called the **stationary distribution** P_∞ of the chain
- It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$



Example: Stationary Distributions

- Question: What's $P(X)$ at time $t = \text{infinity}$?



$$P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain})$$

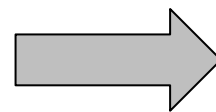
$$P_{\infty}(\text{sun}) = 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 3P_{\infty}(\text{rain})$$

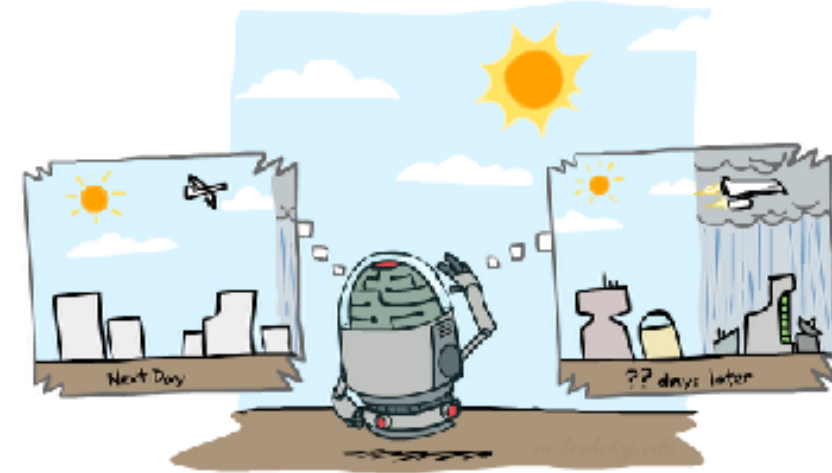
$$P_{\infty}(\text{rain}) = 1/3P_{\infty}(\text{sun})$$

Also: $P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$



$$P_{\infty}(\text{sun}) = 3/4$$

$$P_{\infty}(\text{rain}) = 1/4$$

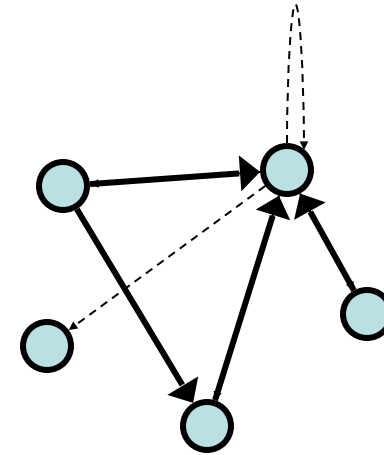


X_{t-1}	X_t	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Application of Stationary Distribution: Web Link Analysis

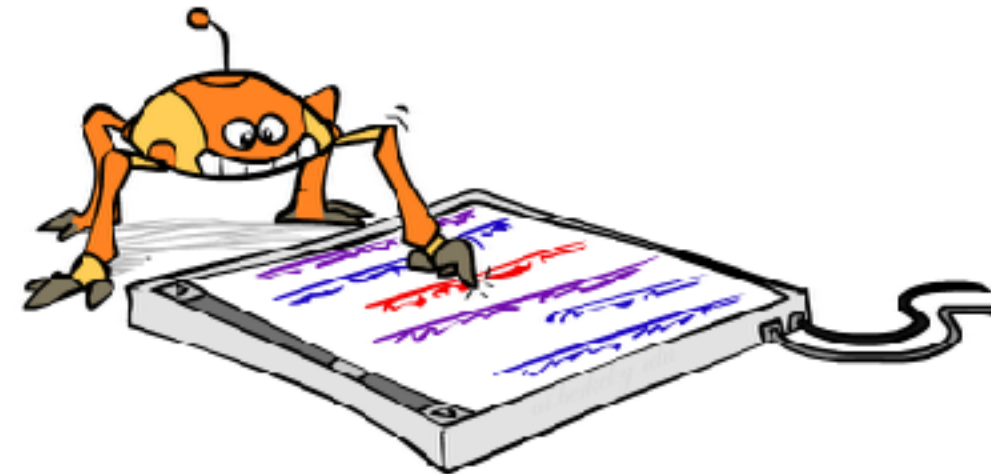
- PageRank over a web graph

- Each web page is a state
- Initial distribution: uniform over pages
- Transitions:
 - With prob. c , uniform jump to a random page (dotted lines, not all shown)
 - With prob. $1-c$, follow a random outlink (solid lines)



- Stationary distribution

- Will spend more time on highly reachable pages
- E.g. many ways to get to the Acrobat Reader download page
- Somewhat robust to link spam (Why?)
- Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



Application of Stationary Distributions: Gibbs Sampling*

- Each joint instantiation over all hidden and query variables is a state: $\{X_1, \dots, X_n\} = H \cup Q$

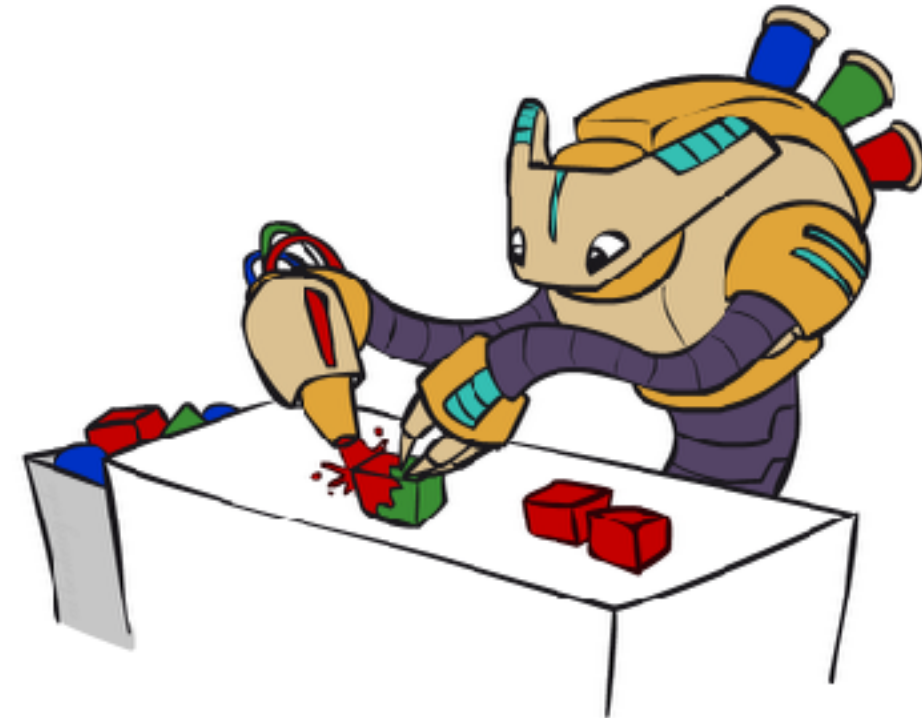
- Transitions:

- With probability $1/n$ resample variable X_j according to

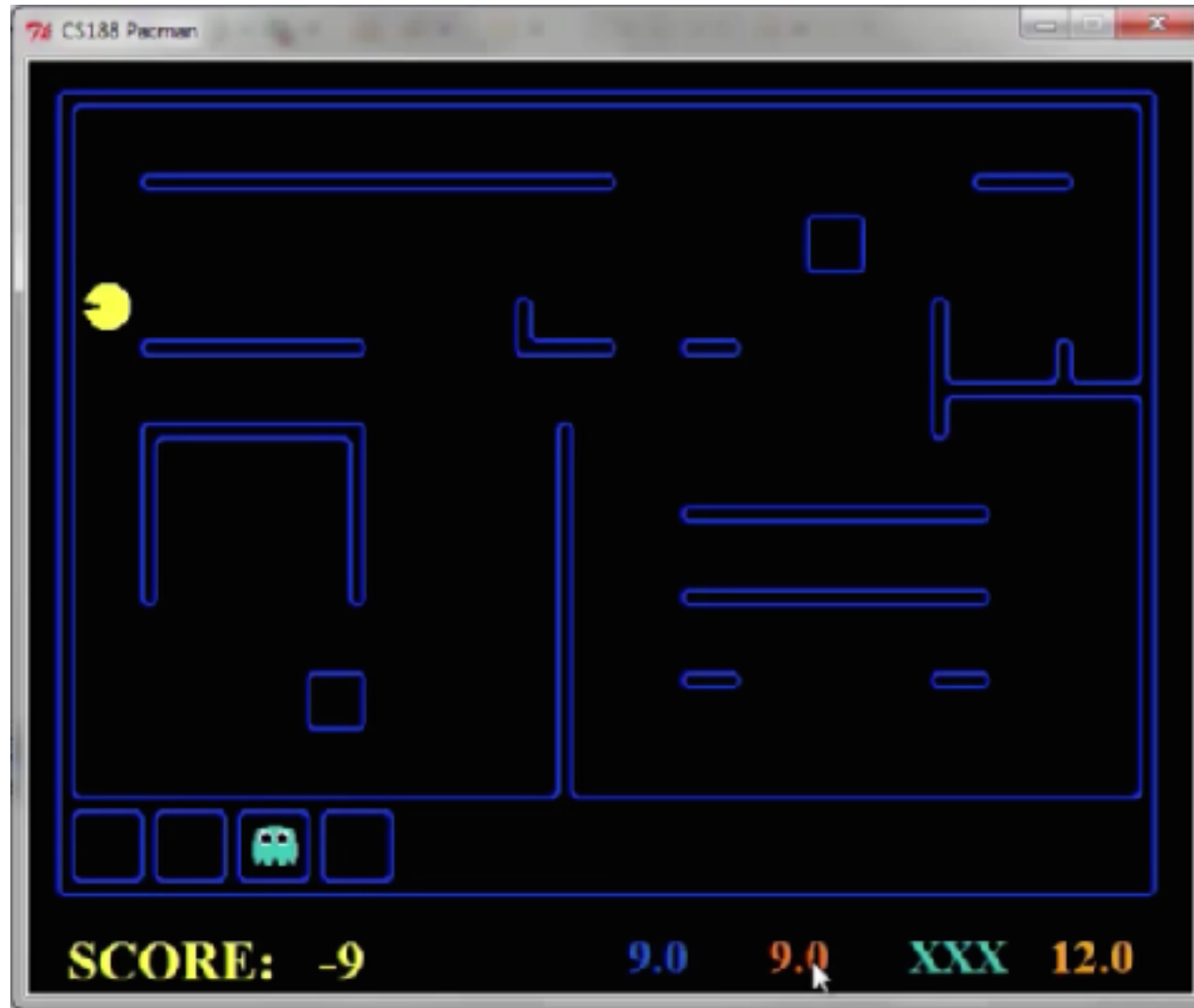
$$P(X_j \mid x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, e_1, \dots, e_m)$$

- Stationary distribution:

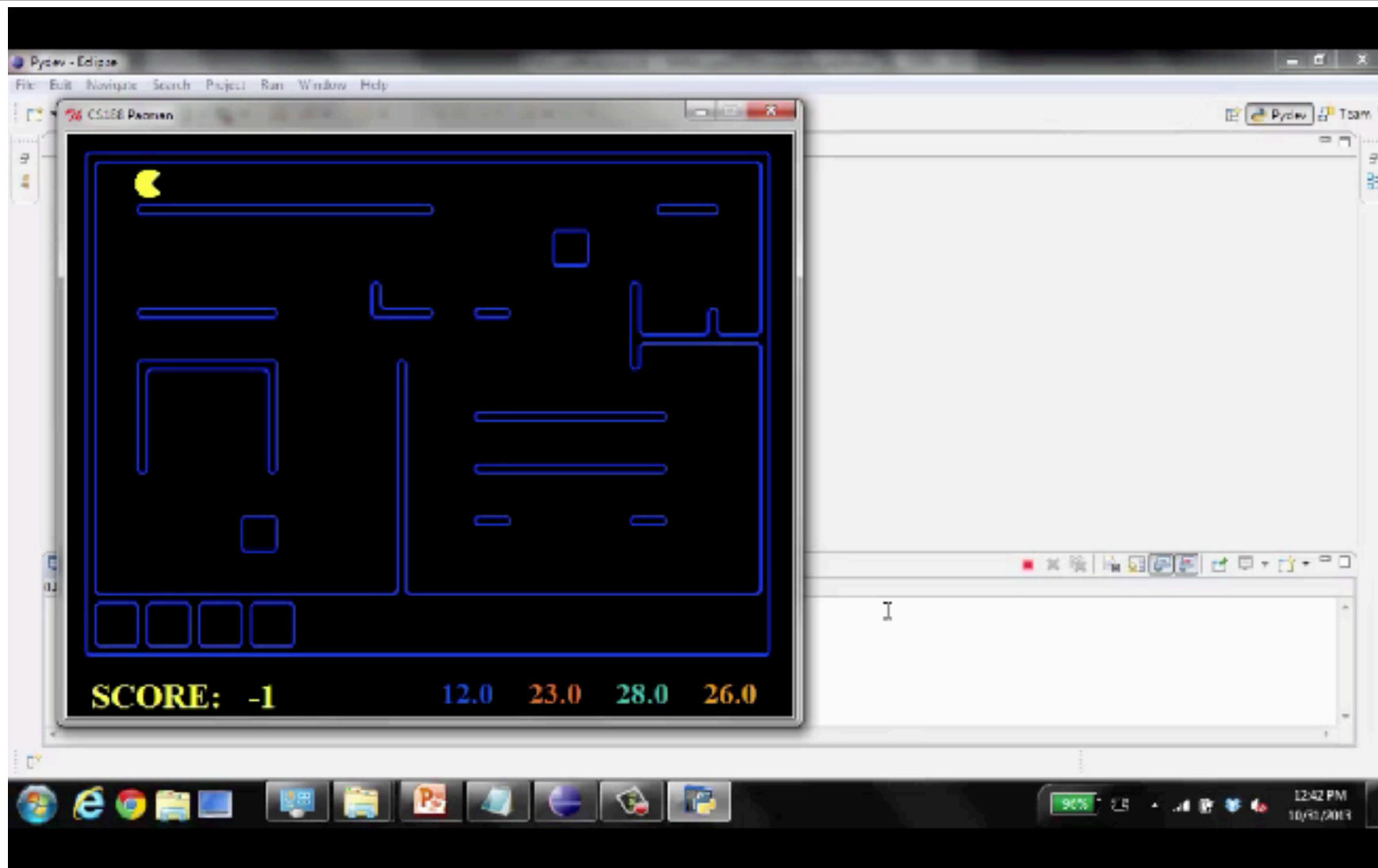
- Conditional distribution $P(X_1, X_2, \dots, X_n \mid e_1, \dots, e_m)$
 - Means that when running Gibbs sampling long enough we get a sample from the desired distribution
 - Requires some proof to show this is true!



Pacman – Sonar (P4)



Pacman – Sonar (no beliefs)

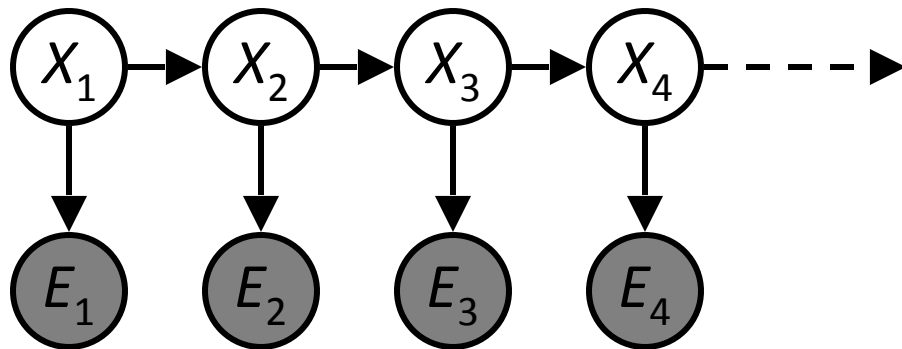


Hidden Markov Models

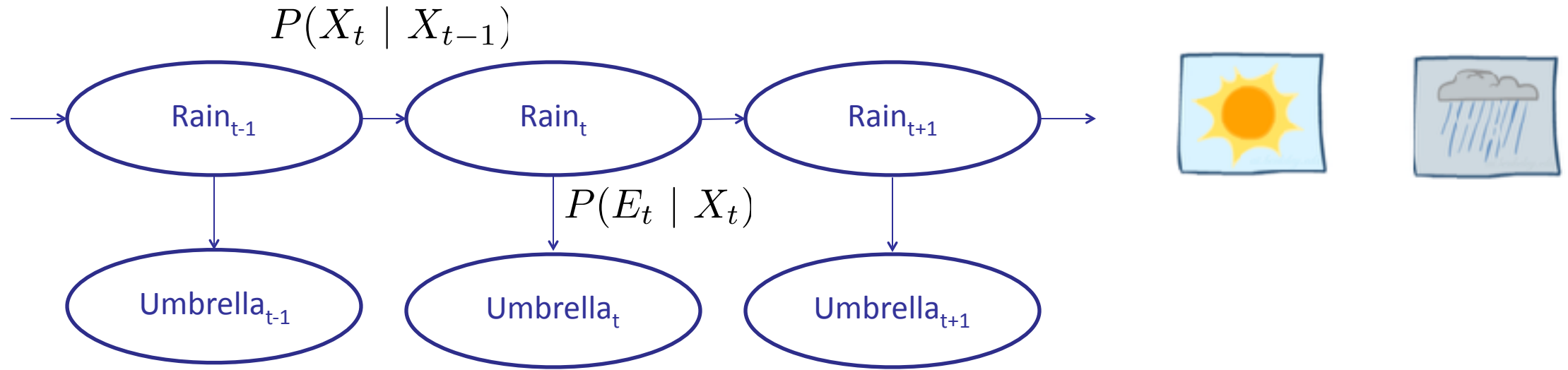


Hidden Markov Models

- Markov chains not so useful for most agents
 - Need observations to update your beliefs
- Hidden Markov models (HMMs)
 - Underlying Markov chain over states X
 - You observe outputs (effects) at each time step



Example: Weather HMM



■ An HMM is defined by:

- Initial distribution:
- Transitions:
- Emissions:

$$P(X_1)$$

$$P(X_t | X_{t-1})$$

$$P(E_t | X_t)$$

R_t	R_{t+1}	$P(R_{t+1} R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Example: Ghostbusters HMM

- $P(X_1) = \text{uniform}$
- $P(X|X') = \text{usually move clockwise, but sometimes move in a random direction or stay in place}$
- $P(R_{ij}|X) = \text{same sensor model as before: red means close, green means far away.}$



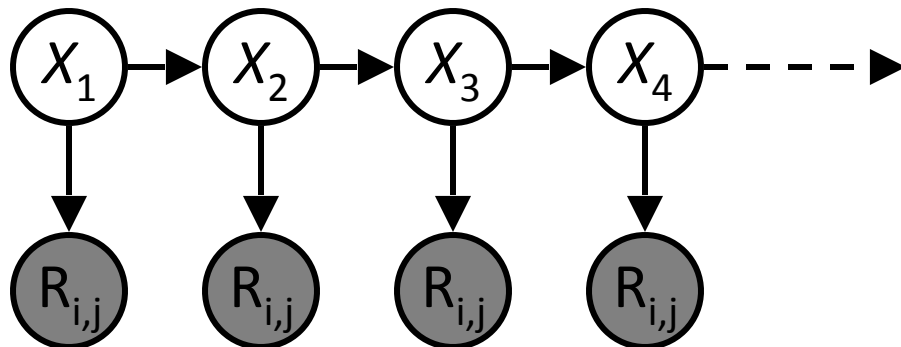
1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$



1/6	1/6	1/2
0	1/6	0
0	0	0

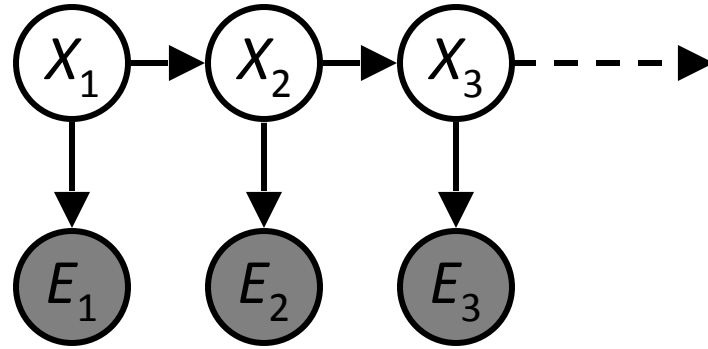
$P(X|X'=\langle 1,2 \rangle)$



Ghostbusters – Circular Dynamics (HMM)



Joint Distribution of an HMM



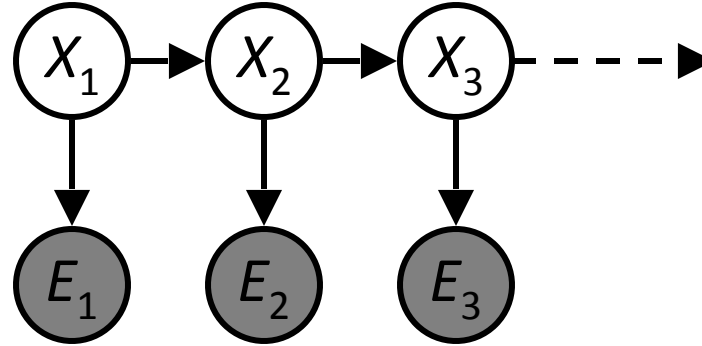
- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

- More generally:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

Implied Conditional Independencies



- Many implied conditional independencies, e.g.,

$$E_1 \perp\!\!\!\perp X_2, E_2, X_3, E_3 \mid X_1$$

- To prove them

- Approach 1: follow similar (algebraic) approach to what we did for Markov models
- Approach 2: D-Separation

Real HMM Examples

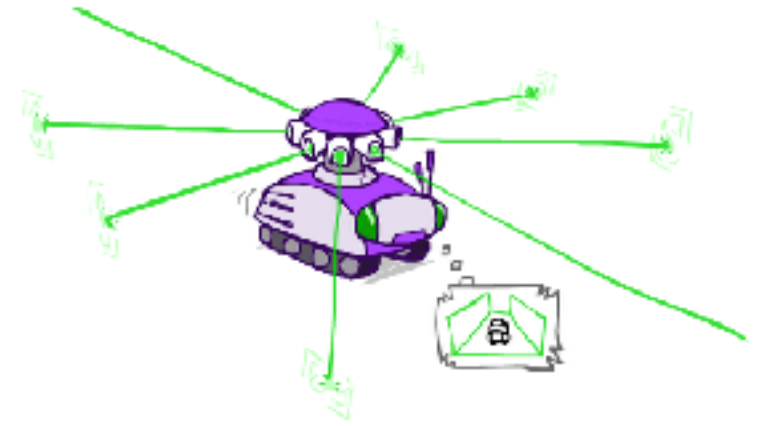
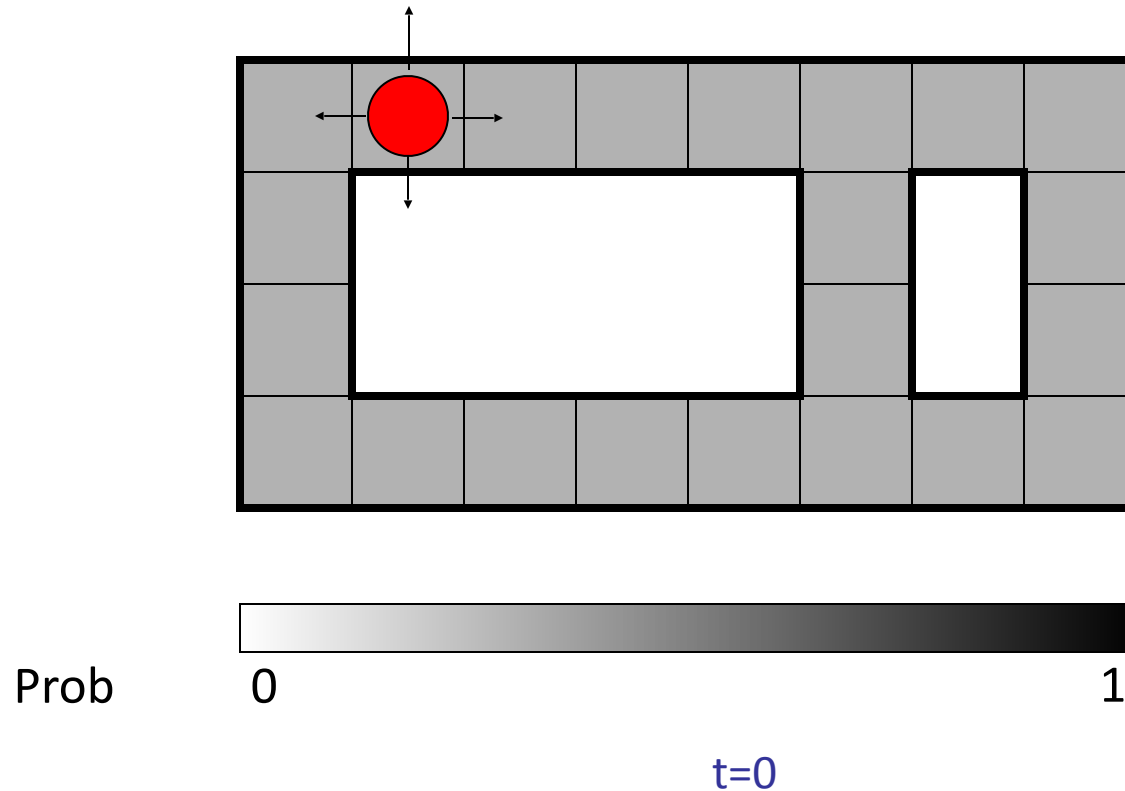
- Speech recognition HMMs:
 - Observations are acoustic signals (continuous valued)
 - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
 - Observations are words (tens of thousands)
 - States are translation options
- Robot tracking:
 - Observations are range readings (continuous)
 - States are positions on a map (continuous)

Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t \mid e_1, \dots, e_t)$ (the belief state) over time
- We start with $B_1(X)$ in an initial setting, usually uniform
- As time passes, or we get observations, we update $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

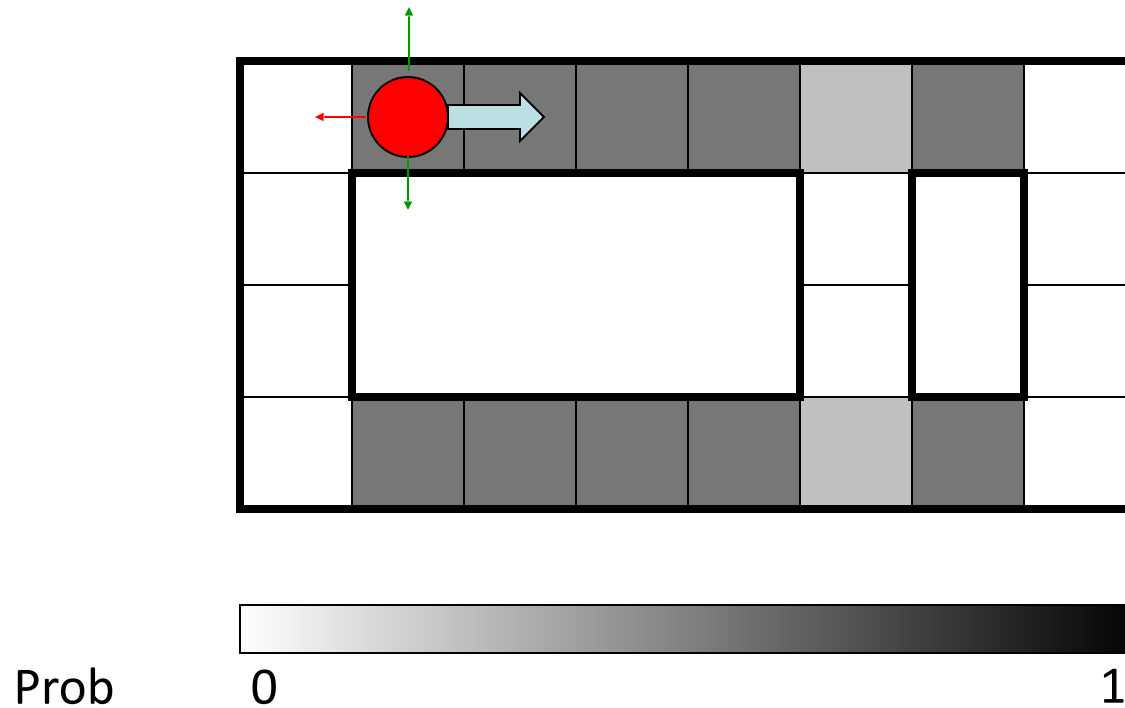
Example: Robot Localization

*Example from
Michael Pfeiffer*



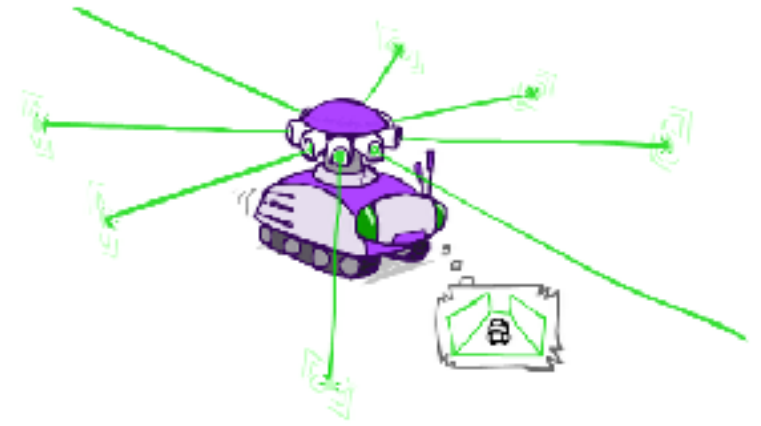
Sensor model: can read in which directions there is a wall, never more than 1 mistake
Motion model: may not execute action with small prob.

Example: Robot Localization

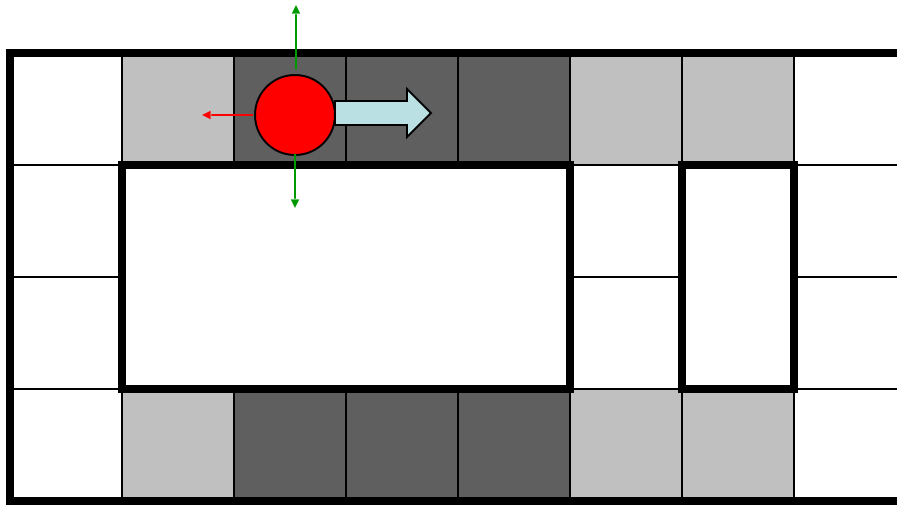


$t=1$

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake



Example: Robot Localization

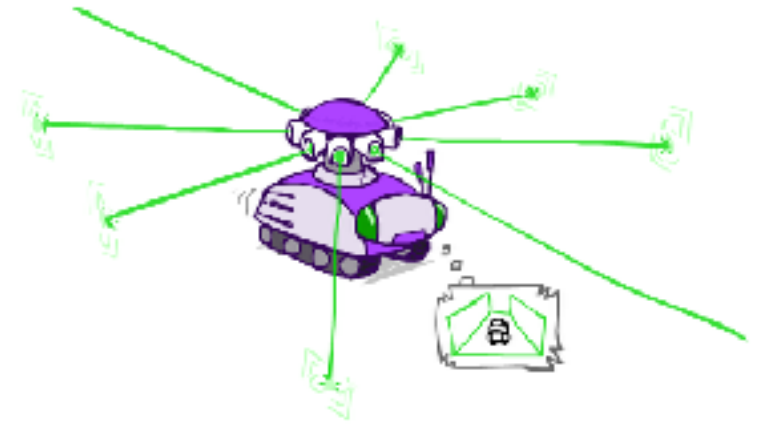


Prob

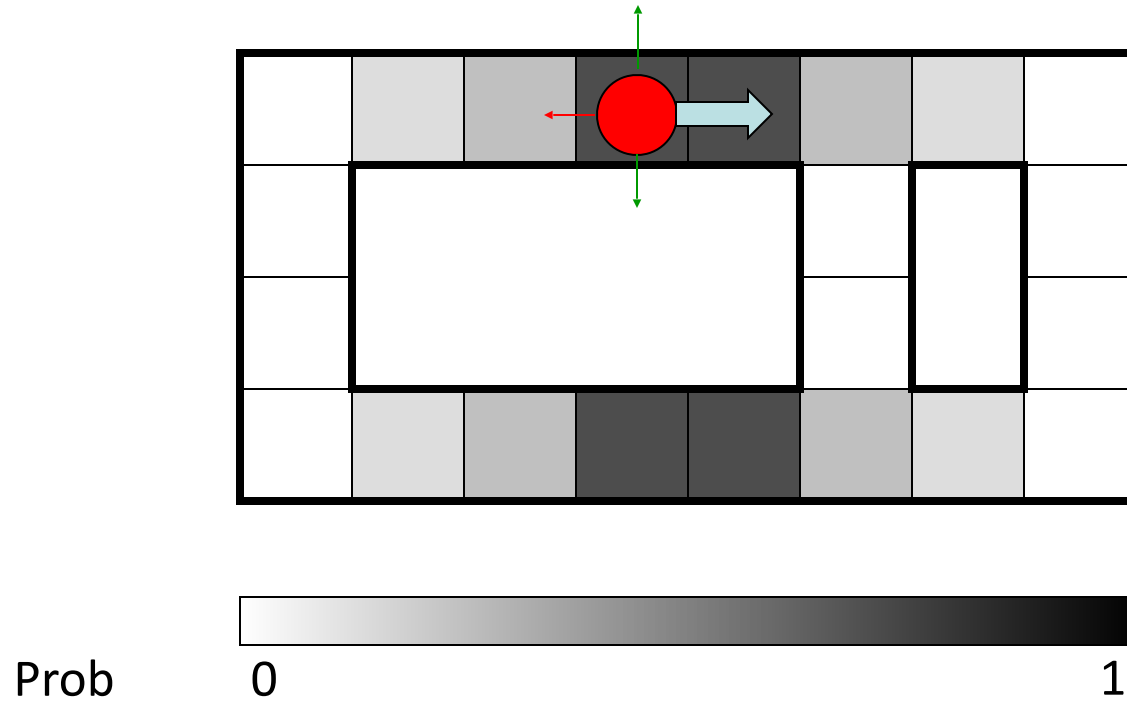
0

1

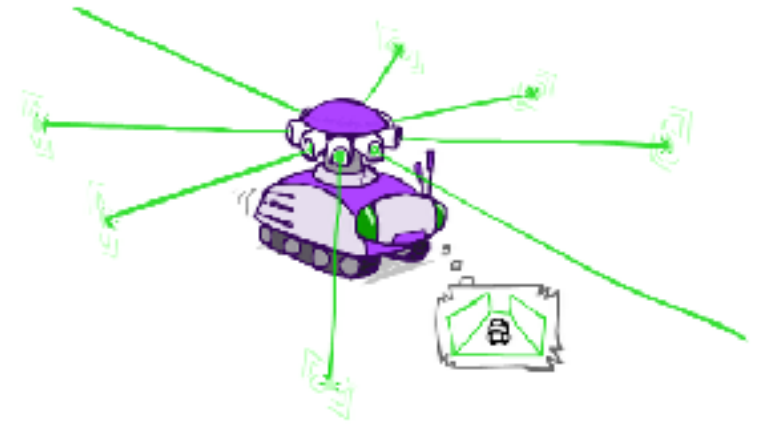
$t=2$



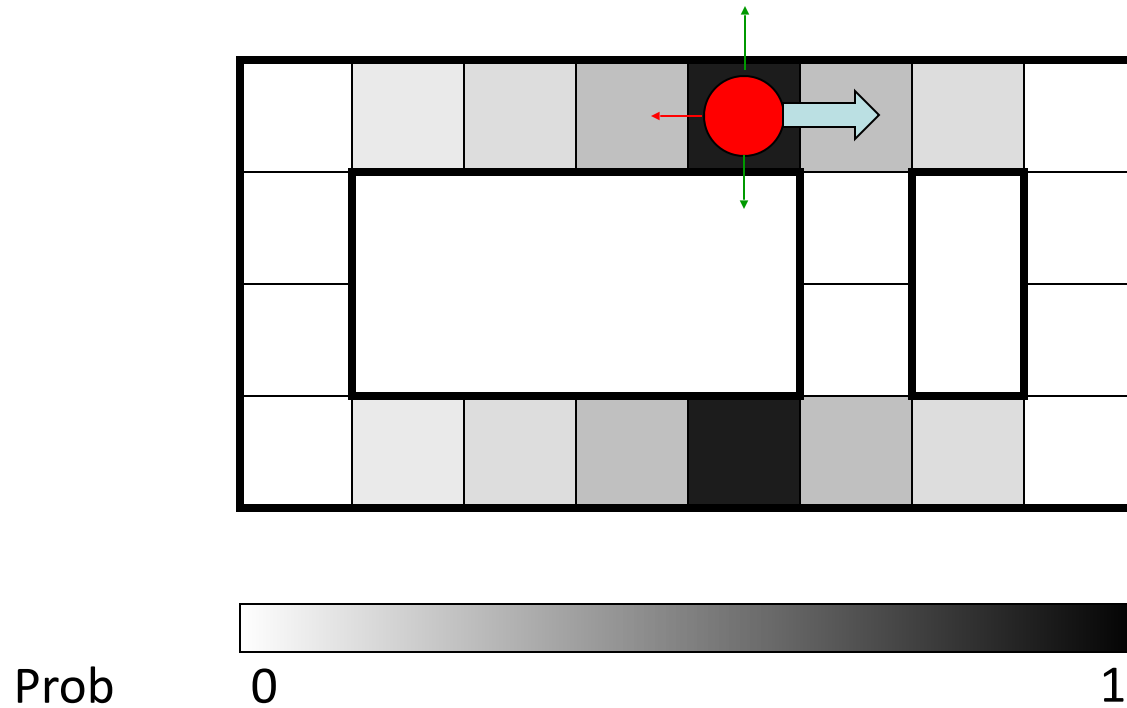
Example: Robot Localization



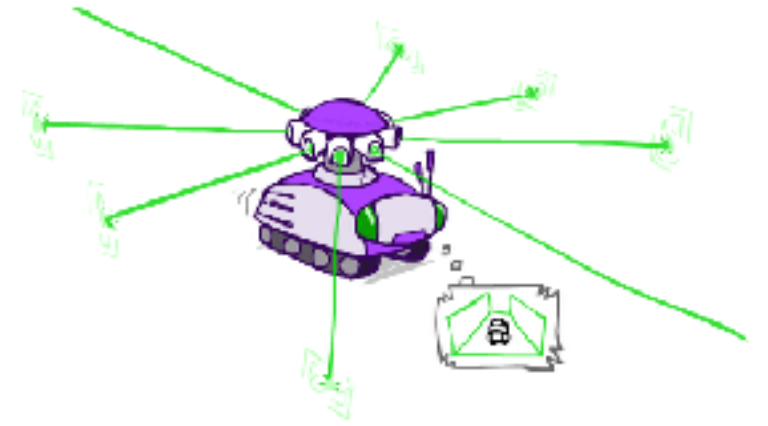
$t=3$

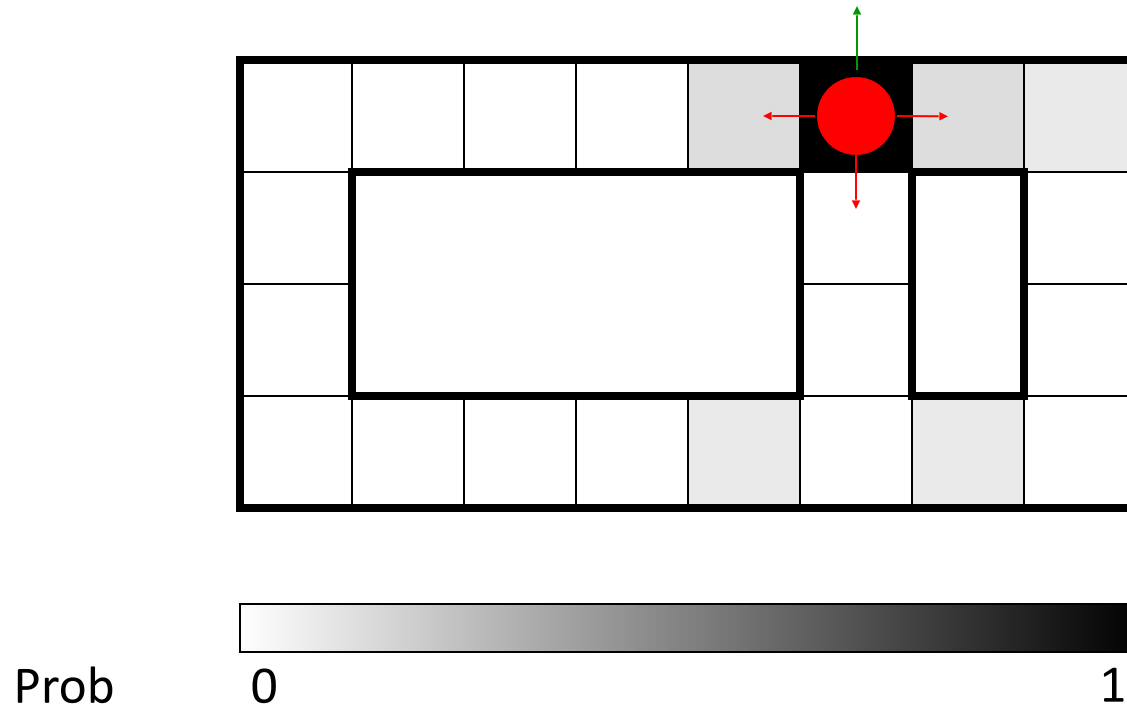


Example: Robot Localization

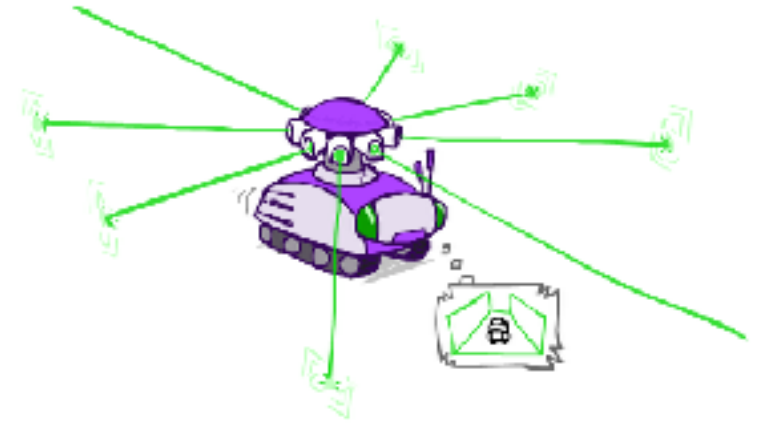


$t=4$





t=5

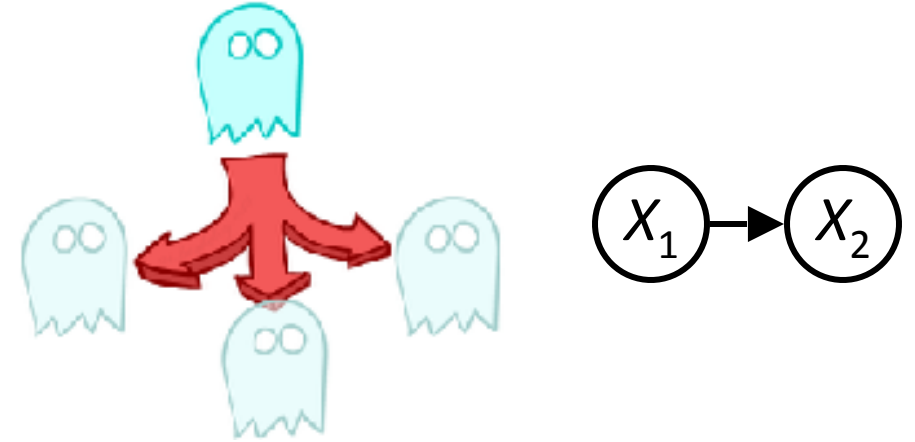


Inference: Base Cases



$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1) / P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1) P(e_1|x_1) \end{aligned}$$



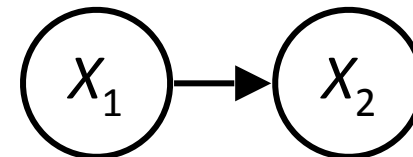
$$P(X_2)$$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1) P(x_2|x_1) \end{aligned}$$

Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$



- Then, after one time step passes:

$$\begin{aligned} P(X_{t+1} | e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t}) \end{aligned}$$

- Or compactly:

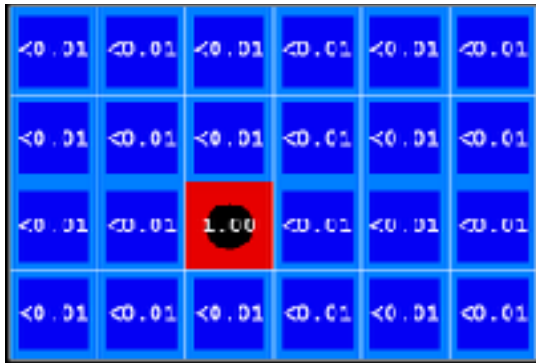
$$B'(X_{t+1}) = \sum_{x_t} P(X' | x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions
 - With the “B” notation, we have to be careful about what time step t the belief is about, and what evidence it includes

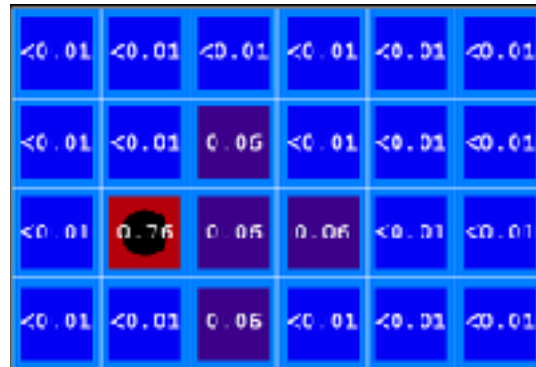
Example: Passage of Time

- As time passes, uncertainty “accumulates”

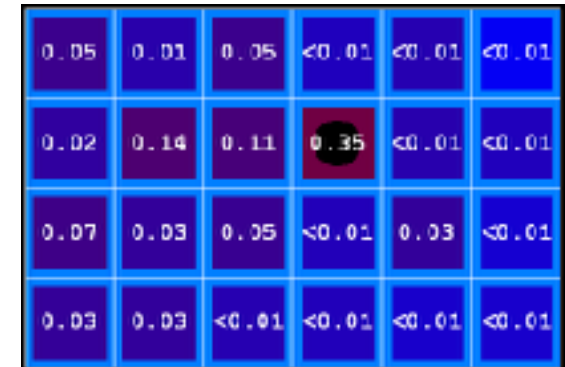
(Transition model: ghosts usually go clockwise)



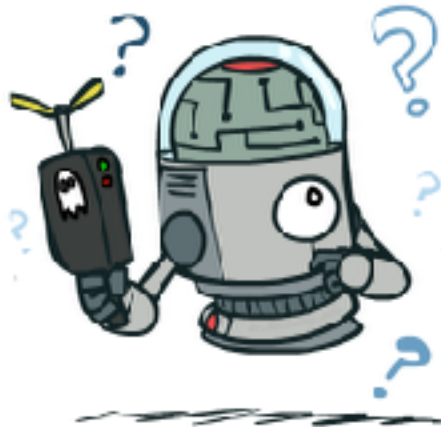
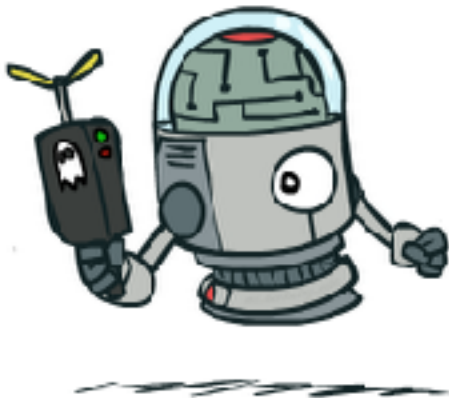
T = 1



T = 2



T = 5



Observation

- Assume we have current belief $P(X \mid \text{previous evidence})$:

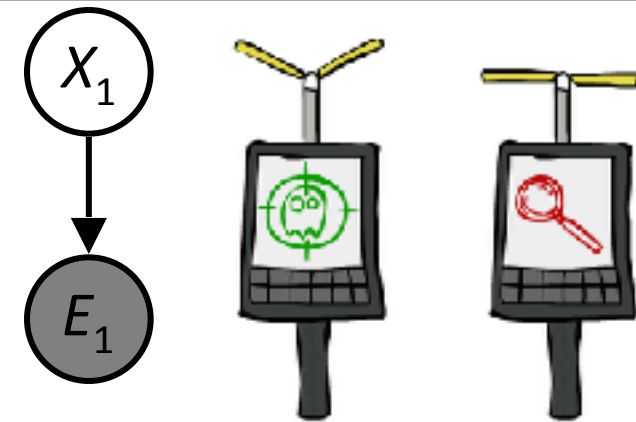
$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then, after evidence comes in:

$$\begin{aligned} P(X_{t+1} | e_{1:t+1}) &= P(X_{t+1}, e_{t+1} | e_{1:t}) / P(e_{t+1} | e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | e_{1:t}, X_{t+1}) P(X_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t}) \end{aligned}$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} | X_{t+1}) B'(X_{t+1})$$



- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

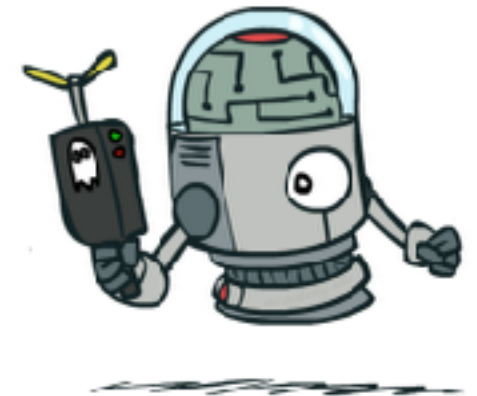
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

$$B(X) \propto P(e|X)B'(X)$$



Putting it All Together: The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t | e_{1:t})$$

- We can derive the following updates

$$\begin{aligned} P(x_t | e_{1:t}) &\propto_X P(x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t) \\ &= P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1}, e_{1:t-1}) \end{aligned}$$

We can normalize as we go if we want to have $P(x|e)$ at each time step, or just once at the end...

Online Belief Updates

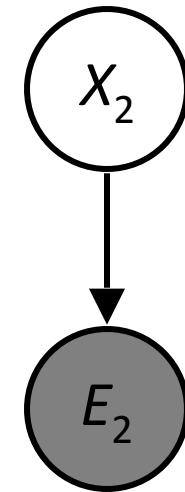
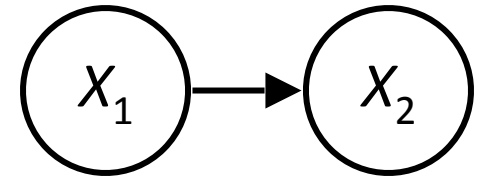
- Every time step, we start with current $P(X \mid \text{evidence})$
- We update for time:

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$

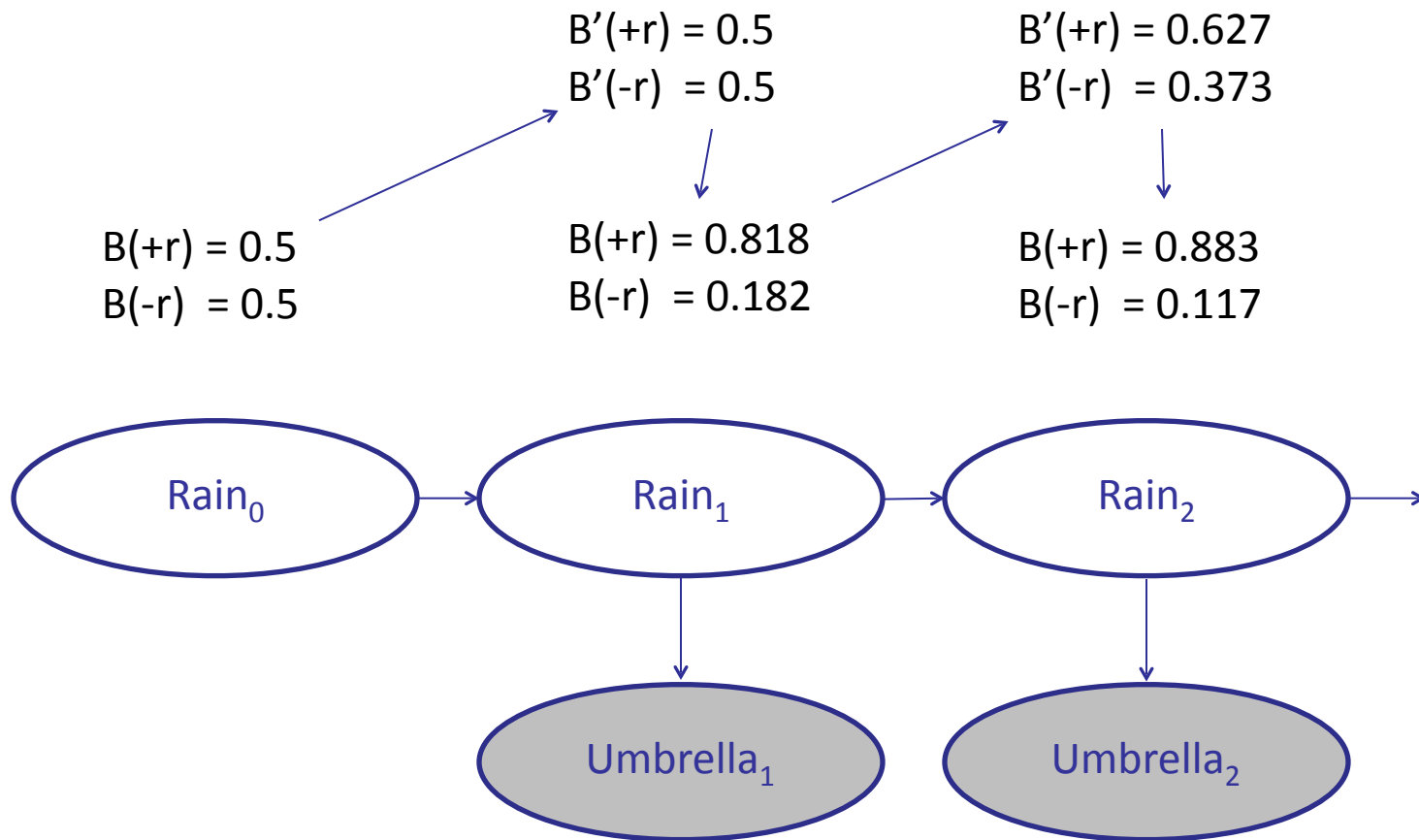
- We update for evidence:

$$P(x_t | e_{1:t}) \propto_X P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$

- The forward algorithm does both at once (and doesn't normalize)



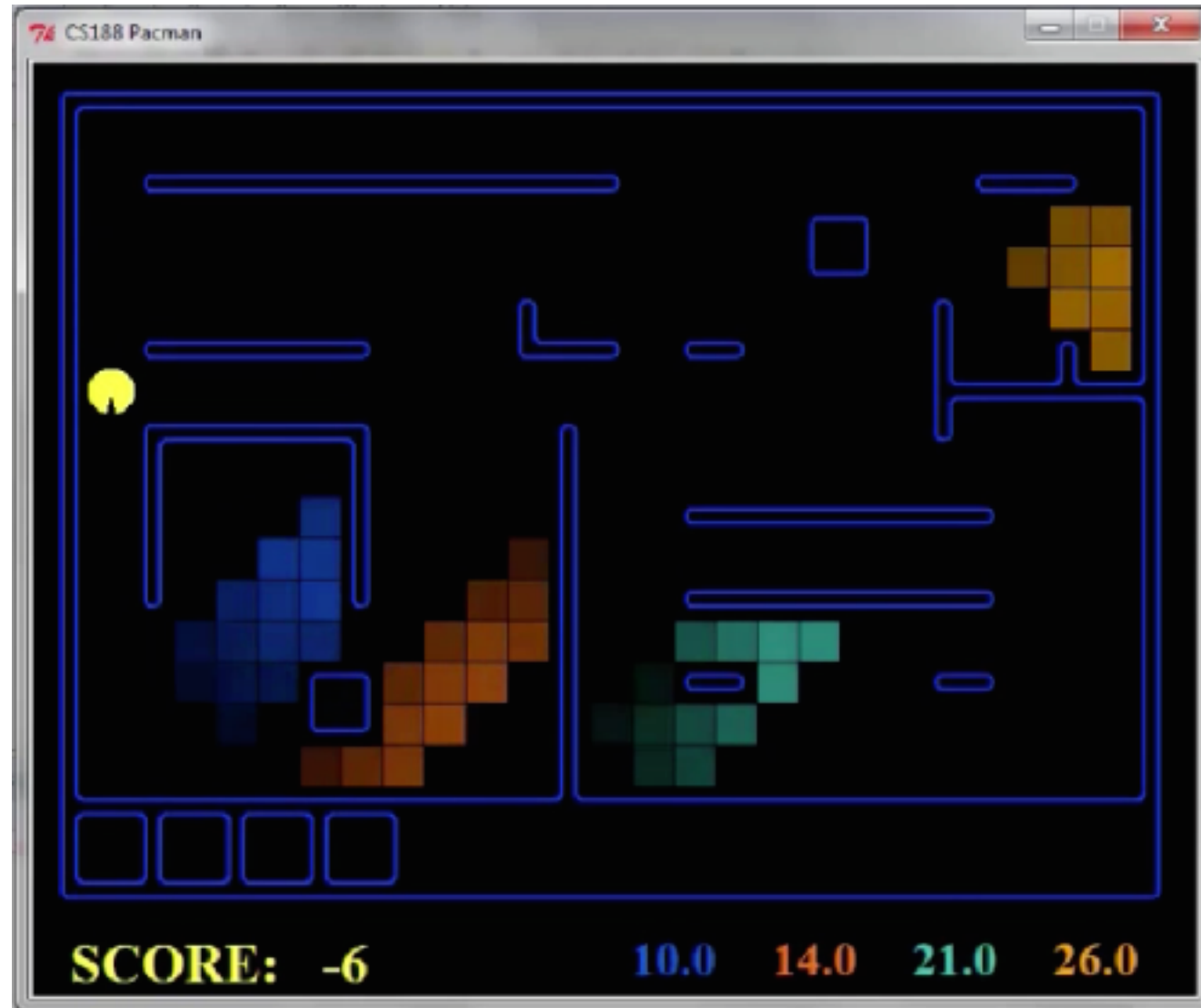
Example: Weather HMM



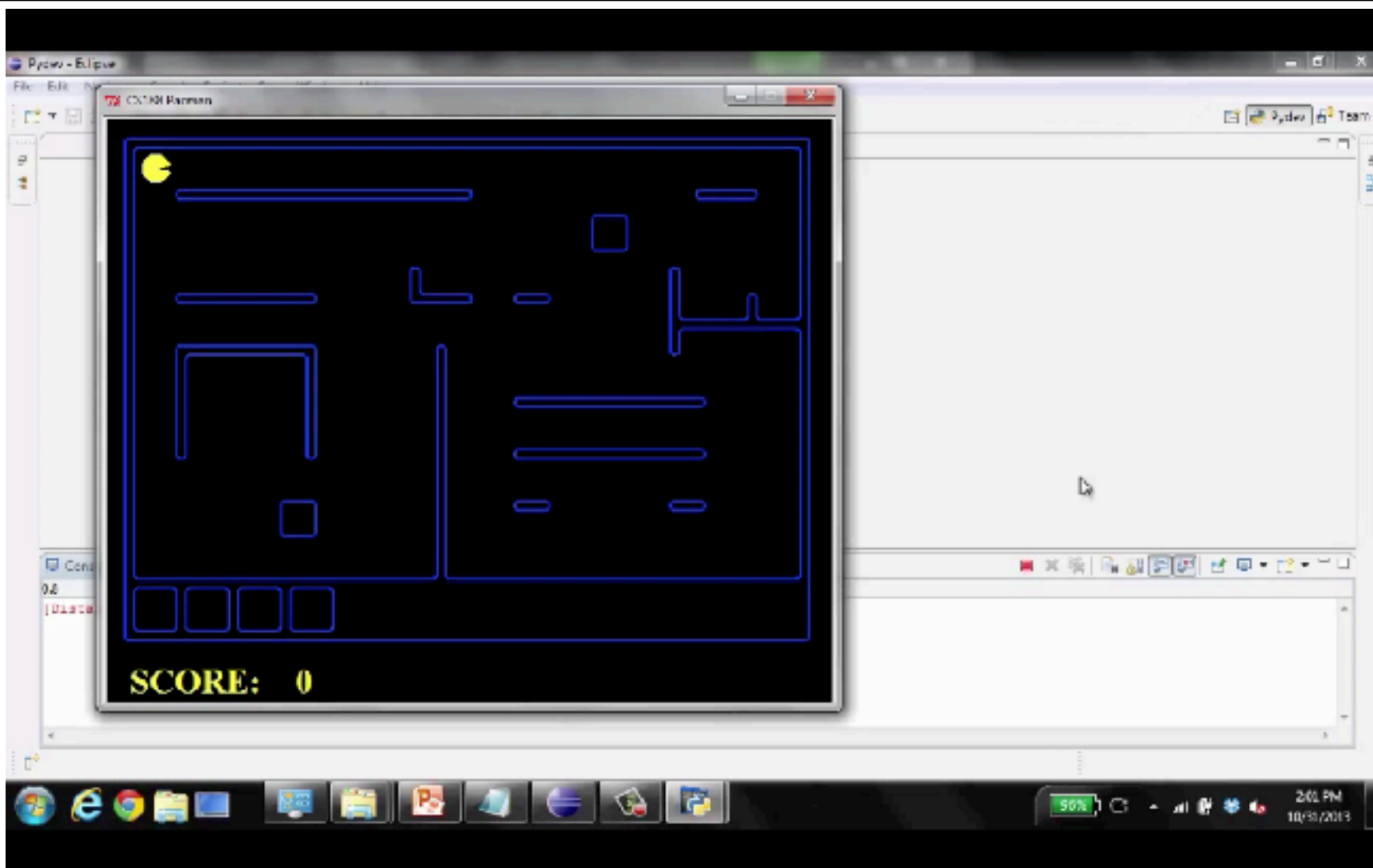
R_t	R_{t+1}	$P(R_{t+1} R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Pacman – Sonar (P4)



Video of Demo Pacman – Sonar (with beliefs)



Next Time: Particle Filtering and Applications of HMMs
