Controlling a Simple System

• Consider a simple system: $\dot{x} = F(x, u)$
  
  – Scalar variables $x$ and $u$, not vectors $x$ and $u$.
  – Assume $x$ is observable: $y = G(x) = x$
  – Assume effect of motor command $u$: $\frac{\partial F}{\partial u} > 0$

• The setpoint $x_{set}$ is the desired value.
  – The controller responds to error: $e = x - x_{set}$

• The goal is to set $u$ to reach $e = 0$. 
The intuitions behind control

• Use action $u$ to push back toward error $e = 0$

• What does pushing back do?
  – Velocity versus acceleration control

• How much should we push back?
  – What does the magnitude of $u$ depend on?
Velocity or acceleration control?

- **Velocity:** \( \dot{x} = (\dot{x}) = F(x, u) = (u) \)
  
  \( x = (x) \)

- **Acceleration:** \( \ddot{x} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = F(x, u) = \begin{pmatrix} v \\ u \end{pmatrix} \)
  
  \( x = \begin{pmatrix} x \\ v \end{pmatrix} \)

  \( \dot{v} = \ddot{x} = u \)
Laws of Motion in Physics

• Newton’s Law: \( F = ma \) or \( a = F/m \).

\[
\begin{pmatrix}
\dot{x} \\
\dot{v}
\end{pmatrix} = \begin{pmatrix}
v \\
F/m
\end{pmatrix}
\]

• But Aristotle said:
  – *Velocity*, not acceleration, is proportional to the force on a body.

• Who is right? Why should we care?
  – (We’ll come back to this.)
The Bang-Bang Controller

- Push back, against the *direction* of the error
- Error: \( e = x - x_{set} \)
  - \( e < 0 \) \( \Rightarrow \) \( u := on \) \( \Rightarrow \) \( \dot{x} = F(x, \text{on}) > 0 \)
  - \( e > 0 \) \( \Rightarrow \) \( u := \text{off} \) \( \Rightarrow \) \( \dot{x} = F(x, \text{off}) < 0 \)
- To prevent chatter around \( e = 0 \)
  - \( e < -\varepsilon \) \( \Rightarrow \) \( u := on \)
  - \( e > +\varepsilon \) \( \Rightarrow \) \( u := \text{off} \)
- Household thermostat. Not very subtle.
Proportional Control

• Push back, *proportional* to the error.

\[ u = -ke + u_b \]

– Set \( u_b \) so that \( \dot{x} = F(x_{set}, u_b) = 0 \)

• For a linear system, exponential convergence.

\[ x(t) = Ce^{-\alpha t} + x_{set} \]

• The controller gain \( k \) determines how quickly the system responds to error.
Velocity Control

• You want the robot to move at velocity $v_{set}$.
• You command velocity $v_{cmd}$.
• You observe velocity $v_{obs}$.

• Define a first-order controller:
  $$\dot{v}_{cmd} = -k (v_{obs} - v_{set})$$
  – $k$ is the controller gain.
Steady-State Offset

• Suppose we have continuing disturbances:
  \[ \dot{x} = F(x,u) + d \]

• The P-controller cannot stabilize at \( e = 0 \).
  – Why not?
Steady-State Offset

• Suppose we have continuing disturbances:

\[
\dot{x} = F(x, u) + d
\]

• The P-controller cannot stabilize at \( e = 0 \).
  – If \( u_b \) is defined so \( F(x_{set}, u_b) = 0 \)
  – then \( F(x_{set}, u_b) + d \neq 0 \), so the system is unstable

• Must adapt \( u_b \) to different disturbances \( d \).
Nonlinear P-control

- Generalize proportional control to
  \[ u = -f(e) + u_b \quad \text{where} \quad f \in M_0^+ \]

- Nonlinear control laws have advantages
  - \( f \) has vertical asymptote: bounded error \( e \)
  - \( f \) has horizontal asymptote: bounded effort \( u \)
  - Possible to converge in finite time.
  - Nonlinearity allows more kinds of composition.
Stopping Controller

- Desired stopping point: $x = 0$.
  - Current position: $x$
  - Distance to obstacle: $d = |x| + \varepsilon$

- Simple P-controller: $v = \dot{x} = -f(x)$

- Finite stopping time for $f(x) = k \sqrt{|x|} \text{sgn}(x)$
Derivative Control

- Damping friction is a force opposing motion, proportional to velocity.
- Try to prevent overshoot by damping controller response.
  \[ u = -k_p e - k_D \dot{e} \]
- Estimating a derivative from measurements is fragile, and amplifies noise.
Adaptive Control

• Sometimes one controller isn’t enough.
• We need controllers at different time scales.

\[ u = -k_p e + u_b \]
\[ \dot{u}_b = -k_i e \quad \text{where} \quad k_i \ll k_p \]

• This can eliminate steady-state offset.
  – Why?
Adaptive Control

• Sometimes one controller isn’t enough.
• We need controllers at different time scales.
  \[ u = -k_p e + u_b \]
  \[ \dot{u}_b = -k_i e \quad \text{where} \quad k_i \ll k_p \]
• This can eliminate steady-state offset.
  – Because the slower controller adapts \( u_b \).
Integral Control

• The adaptive controller $\dot{u}_b = -k_I e$ means
  
  $$u_b(t) = -k_I \int_0^t e \, dt + u_b$$

• Therefore
  
  $$u(t) = -k_P e(t) - k_I \int_0^t e \, dt + u_b$$

• The Proportional-Integral (PI) Controller.
The PID Controller

- A weighted combination of Proportional, Integral, and Derivative terms.
  \[ u(t) = -k_p e(t) - k_i \int_0^t e \, dt - k_d \dot{e}(t) \]

- The PID controller is the workhorse of the control industry. Tuning is non-trivial.
  - Next lecture includes some tuning methods.
Habituation

- Integral control adapts the bias term $u_b$.
- Habituation adapts the setpoint $x_{set}$.
  - It prevents situations where too much control action would be dangerous.
- Both adaptations reduce steady-state error.

\[ u = -k_p e + u_b \]

\[ \dot{x}_{set} = +k_h e \quad \text{where} \quad k_h \ll k_p \]
Types of Controllers

- **Feedback control**
  - Sense error, determine control response.

- **Feedforward control**
  - Sense disturbance, predict resulting error, respond to predicted error before it happens.

- **Model-predictive control**
  - Plan trajectory to reach goal.
  - Take first step.
  - Repeat.
Laws of Motion in Physics

• Newton’s Law: \( F=ma \) or \( a=F/m \).

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\begin{bmatrix}
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\end{bmatrix}
\]

• But Aristotle said:
  – \textit{Velocity}, not acceleration, is proportional to the force on a body.

• Who is right? Why should we care?
Who is right? Aristotle!

- Try it! It takes constant force to keep an object moving at constant velocity.
  - Ignore brief transients
- Aristotle was a genius to recognize that there could be laws of motion, and to formulate a useful and accurate one.
- This law is true because our everyday world is friction-dominated.
Who is right? Newton!

• Newton’s genius was to recognize that the true laws of motion may be different from what we usually observe on earth.

• For the planets, without friction, motion continues without force.

• For Aristotle, “force” means $F_{\text{external}}$.

• For Newton, “force” means $F_{\text{total}}$.
  – On Earth, you must include $F_{\text{friction}}$. 
From Newton back to Aristotle

- \( F_{\text{total}} = F_{\text{external}} + F_{\text{friction}} \)
- \( F_{\text{friction}} = -f(v) \) for some monotonic \( f \).

Thus:
\[
\begin{pmatrix}
\dot{x} \\
\dot{v}
\end{pmatrix}
= 
\begin{pmatrix}
v \\
F/m
\end{pmatrix}
= 
\begin{pmatrix}
v \\
\frac{1}{m} F_{\text{ext}} - \frac{1}{m} f(v)
\end{pmatrix}
\]

- Velocity \( v \) moves quickly to equilibrium:
\[
\dot{v} = \frac{1}{m} F_{\text{ext}} - \frac{1}{m} f(v)
\]

- Terminal velocity \( v_{\text{final}} \) depends on:
  - \( F_{\text{ext}}, m, \) and the friction function \( f(v) \).
  - So Aristotle was right! In a friction-dominated world.