Lecture 13: Mapping Landmarks

CS 344R: Robotics
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Landmark Map

• Locations and uncertainties of $n$ landmarks, with respect to a specific frame of reference.
  – World frame: fixed origin point
  – Robot frame: origin at the robot

• **Problem**: how to combine new information with old to update the map.
A Spatial Relationship is a Vector

- A spatial relationship holds between two poses: the position and orientation of one, in the frame of reference of the other.

\[
x_{AB} = \begin{bmatrix} x_{AB} \\ y_{AB} \\ \phi_{AB} \end{bmatrix}
\]

\[
x_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T
\]
Uncertain Spatial Relationships

\[ \hat{x} = E[x] \]
\[ C(x) = E[(x - \hat{x})(x - \hat{x})^T] \]

- An uncertain spatial relationship is described by a probability distribution of vectors, with a mean and a covariance matrix.

\[
\hat{x} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{\phi} \end{bmatrix}, \quad C(x) = \begin{bmatrix}
\sigma_x^2 & \sigma_{xy} & \sigma_{x\phi} \\
\sigma_{xy} & \sigma_y^2 & \sigma_{y\phi} \\
\sigma_{x\phi} & \sigma_{y\phi} & \sigma_{\phi}^2 
\end{bmatrix}
\]
A Map with $n$ Landmarks

- Concatenate $n$ vectors into one big state vector
\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]
\[
\hat{x} = \begin{bmatrix}
\hat{x}_1 \\
\hat{x}_2 \\
\vdots \\
\hat{x}_n
\end{bmatrix}
\]

- And one big $3n \times 3n$ covariance matrix.
\[
C(x) = \begin{bmatrix}
C(x_1) & C(x_1, x_2) & \cdots & C(x_1, x_n) \\
C(x_2, x_1) & C(x_2) & \cdots & C(x_2, x_n) \\
\vdots & \vdots & \ddots & \vdots \\
C(x_n, x_1) & C(x_n, x_2) & \cdots & C(x_n)
\end{bmatrix}
\]
Example

- The robot senses object #1.
- The robot moves.
- The robot senses a *different* object #2.
- Now the robot senses object #1 again.

- After each step, what does the robot know (in its landmark map) about each object, including itself?
Robot Senses Object #1 and Moves
Robot Senses Object #2
Robot Senses Object #1 Again
Updated Estimates After Constraint
Compounding

\[ \mathbf{x}_{BC} = \begin{bmatrix} x_{BC} & y_{BC} & \phi_{BC} \end{bmatrix}^T \]

\[ \mathbf{x}_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T \]
Compounding

\[ \mathbf{x}_{AC} = \begin{bmatrix} x_{AC} & y_{AC} & \phi_{AC} \end{bmatrix}^T \]

\[ \mathbf{x}_{BC} = \begin{bmatrix} x_{BC} & y_{BC} & \phi_{BC} \end{bmatrix}^T \]

\[ \mathbf{x}_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T \]
Rotation Matrix

\[
\begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
x_G \\
y_G
\end{bmatrix} =
\begin{bmatrix}
x_B \\
y_B
\end{bmatrix}
\]
Compounding

• Let $x_{AB}$ be the pose of object B in the frame of reference of A. (Sometimes written $B_A$.)

• Given $x_{AB}$ and $x_{BC}$, calculate $x_{AC}$.

$$x_{AC} = x_{AB} \oplus x_{BC} = \begin{bmatrix} x_{BC} \cos \phi_{AB} - y_{BC} \sin \phi_{AB} + x_{AB} \\ x_{BC} \sin \phi_{AB} + y_{BC} \cos \phi_{AB} + y_{AB} \\ \phi_{AB} + \phi_{BC} \end{bmatrix}$$

• Compute $C(x_{AC})$ from $C(x_{AB})$, $C(x_{BC})$, and $C(x_{AB}, x_{BC})$. 
Computing Covariance

- Consider the linear mapping \( y = Mx + b \)

\[
y = Mx + b \\
C(y) = C(Mx + b) \\
= \mathbb{E}[(Mx + b - (M\hat{x} + b)) (Mx + b - (M\hat{x} + b))^T] \\
= \mathbb{E}[M(x - \hat{x}) (M(x - \hat{x}))^T] \\
= \mathbb{E}[M (x - \hat{x})(x - \hat{x})^T M^T] \\
= M \mathbb{E}[(x - \hat{x})(x - \hat{x})^T] M^T \\
= MC(x) M^T
\]

- Apply this to nonlinear functions by using Taylor Series.
Inverse Relationship

\[ \mathbf{x}_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T \]

\[ \mathbf{x}_{BA} = \begin{bmatrix} x_{BA} & y_{BA} & \phi_{BA} \end{bmatrix}^T \]
The Inverse Relationship

• Let $x_{AB}$ be the pose of object B in the frame of reference of A.

• Given $x_{AB}$, calculate $x_{BA}$.

\[
x_{BA} = (-)x_{AB} = \begin{bmatrix}
-x_{AB} \cos \phi_{AB} - y_{AB} \sin \phi_{AB} \\
x_{AB} \sin \phi_{AB} - y_{AB} \cos \phi_{AB} \\
-\phi_{AB}
\end{bmatrix}
\]

• Compute $C(x_{BA})$ from $C(x_{AB})$
Composite Relationships

• Compounding combines relationships head-to-tail: \( x_{AC} = x_{AB} \oplus x_{BC} \)

• Tail-to-tail combinations come from observing two things from the same point:
  \( x_{BC} = (\ominus x_{AB}) \oplus x_{AC} \)

• Head-to-head combinations come from two observations of the same thing:
  \( x_{AC} = x_{AB} \oplus (\ominus x_{CB}) \)

• They provide new relationships between their endpoints.
Merging Information

• An uncertain observation of a pose is combined with previous knowledge using the extended Kalman filter.
  – Previous knowledge: $x_k^-, P_k$
  – New observation: $z_k, R$

• Update: $x = x(\text{new}) \otimes x(\text{old})$

• Can integrate dynamics as well.
EKF Update Equations

• Predictor step: \[ \hat{x}_k^- = f(\hat{x}_{k-1}, u_k) \]
  \[ P_k^- = A P_{k-1} A^T + Q \]

• Kalman gain: \[ K_k = P_k^- H^T (H P_k^- H^T + R)^{-1} \]

• Corrector step: \[ \hat{x}_k = \hat{x}_k^- + K_k (z_k - h(\hat{x}_k^-)) \]
  \[ P_k = (I - K_k H) P_k^- \]
Reversing and Compounding

- \( OBJ1_R = (\ominus ROBOT_W) \oplus OBJ1_W \)
- \( = WORLD_R \oplus OBJ1_W \)
Sensing Object #2

- $\text{OBJ2}_w = \text{ROBOT}_w \oplus \text{OBJ2}_R$
Observing Object #1 Again

- \( \text{OBJ}_1^W = \text{ROBOT}_W \oplus \text{OBJ}_1^R \)
Combining Observations (1)

- $OJB_1^W = OJB_1^W(\text{new}) \otimes \text{OBJ}1^W(\text{old})$
- $OJB_1^R = OJB_1^R(\text{new}) \otimes \text{OBJ}1^R(\text{old})$
Combining Observations (2)

- ROBOT_{W}(new) = OBJ_{1W} \oplus (\ominus OBJ_{1R})
- ROBOT_{W} = ROBOT_{W}(new) \otimes ROBOT_{W}(old)
Useful for Feature-Based Maps

• We’ll see this again when we study FastSLAM.