Lecture 13:
Mapping Landmarks

CS 344R: Robotics

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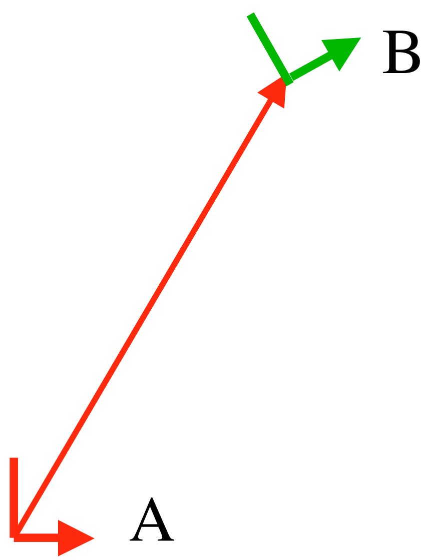
Landmark Map

- Locations and uncertainties of n landmarks, with respect to a specific frame of reference.
  - World frame: fixed origin point
  - Robot frame: origin at the robot

- **Problem**: how to combine new information with old to update the map.

A Spatial Relationship is a Vector

- A spatial relationship holds between two poses: the position and orientation of one, in the frame of reference of the other.



$$\mathbf{x}_{AB} = \begin{bmatrix} x_{AB} \\ y_{AB} \\ \phi_{AB} \end{bmatrix}$$

$$\mathbf{x}_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T$$

Uncertain Spatial Relationships

$$\hat{\mathbf{x}} = E[\mathbf{x}]$$

$$\mathbf{C}(\mathbf{x}) = E[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T]$$

- An *uncertain* spatial relationship is described by a probability distribution of vectors, with a *mean* and a *covariance matrix*.

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{\phi} \end{bmatrix} \qquad \mathbf{C}(\mathbf{x}) = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x\phi} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y\phi} \\ \sigma_{x\phi} & \sigma_{y\phi} & \sigma_\phi^2 \end{bmatrix}$$

A Map with *n* Landmarks

- Concatenate *n* vectors into one big state vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} \qquad \hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \vdots \\ \hat{\mathbf{x}}_n \end{bmatrix}$$

- And one big $3n \times 3n$ covariance matrix.

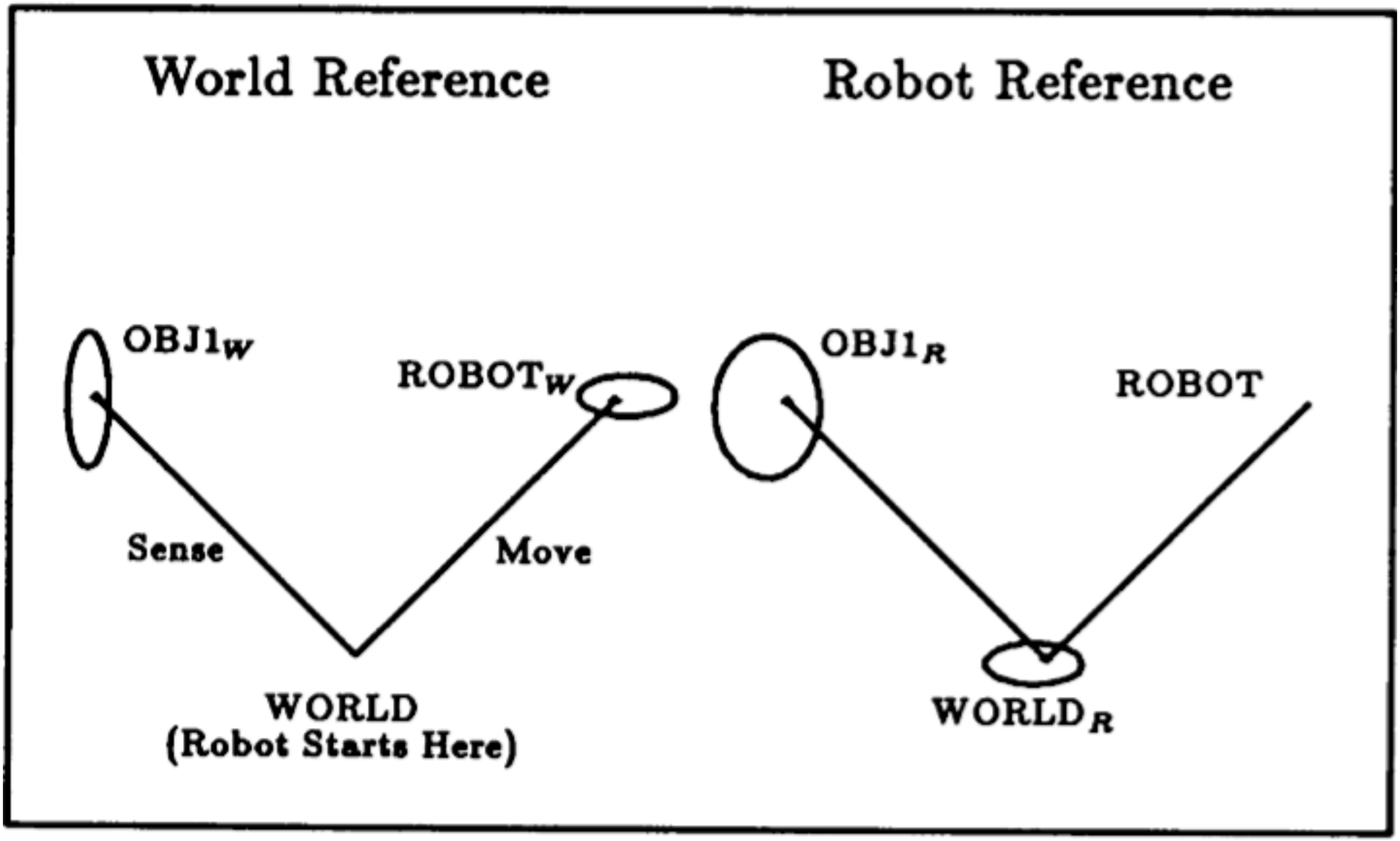
$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} \mathbf{C}(\mathbf{x}_1) & \mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) & \cdots & \mathbf{C}(\mathbf{x}_1, \mathbf{x}_n) \\ \mathbf{C}(\mathbf{x}_2, \mathbf{x}_1) & \mathbf{C}(\mathbf{x}_2) & \cdots & \mathbf{C}(\mathbf{x}_2, \mathbf{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}(\mathbf{x}_n, \mathbf{x}_1) & \mathbf{C}(\mathbf{x}_n, \mathbf{x}_2) & \cdots & \mathbf{C}(\mathbf{x}_n) \end{bmatrix}$$

Example

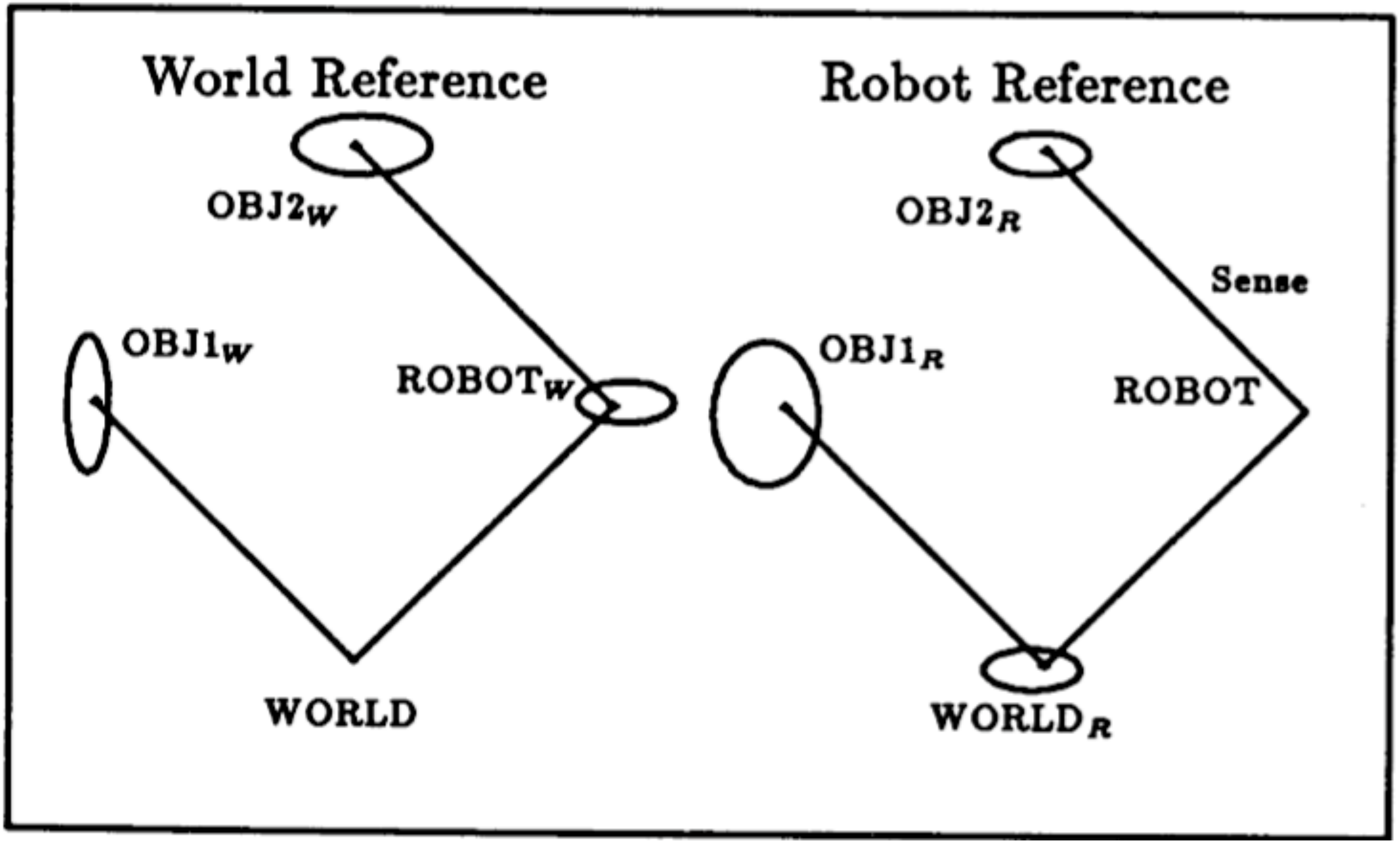
- The robot senses object #1.
- The robot moves.
- The robot senses a *different* object #2.
- Now the robot senses object #1 again.

- After each step, what does the robot know (in its landmark map) about each object, including itself?

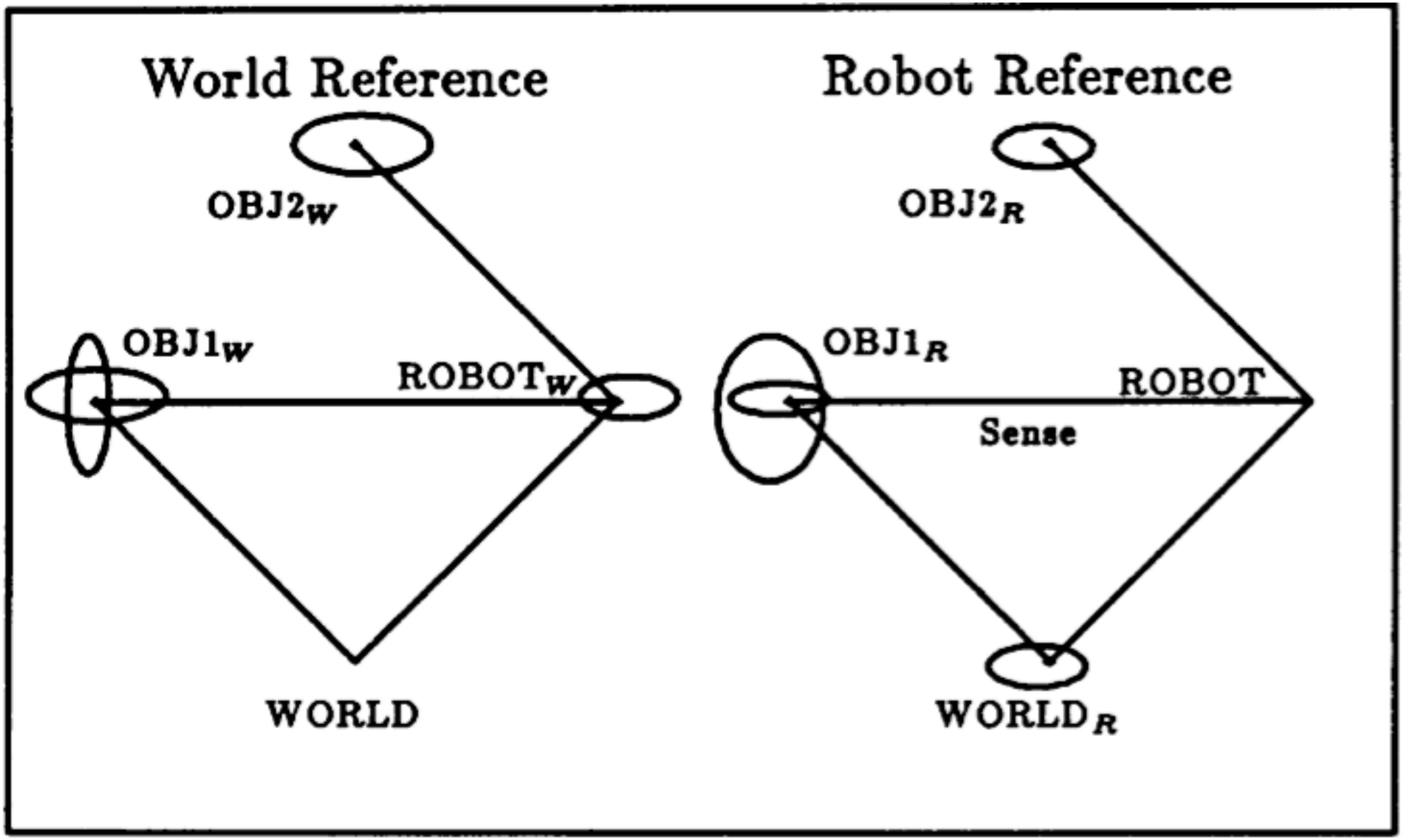
Robot Senses Object #1 and Moves



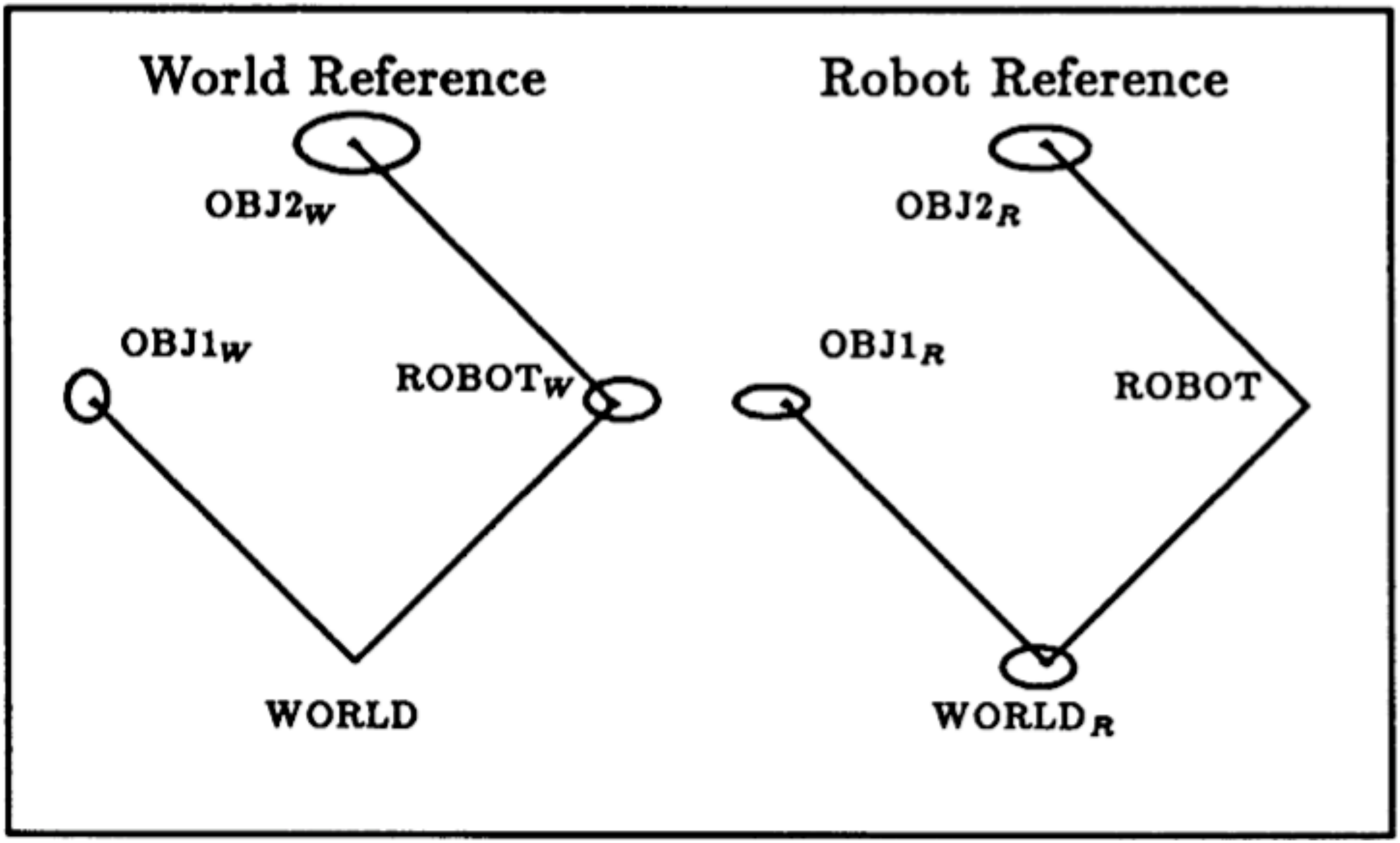
Robot Senses Object #2



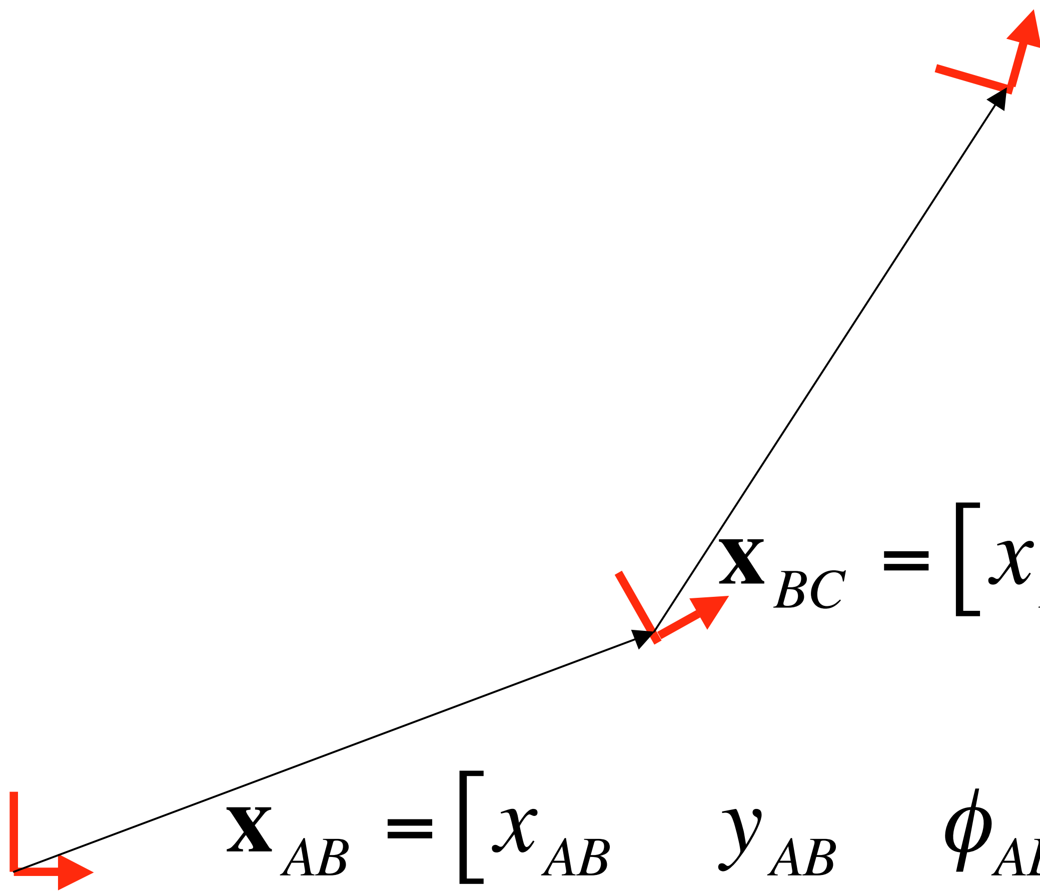
Robot Senses Object #1 Again



Updated Estimates After Constraint



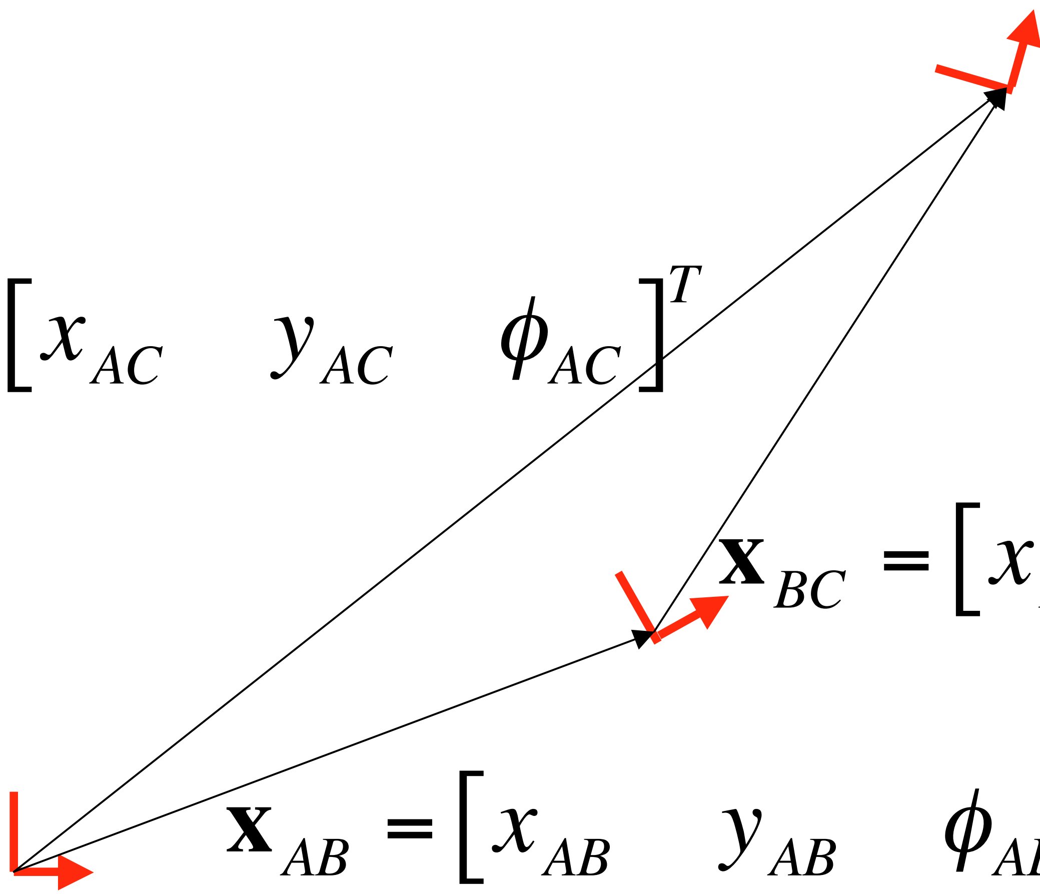
Compounding



$$\mathbf{x}_{BC} = \begin{bmatrix} x_{BC} & y_{BC} & \phi_{BC} \end{bmatrix}^T$$

$$\mathbf{x}_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T$$

Compounding



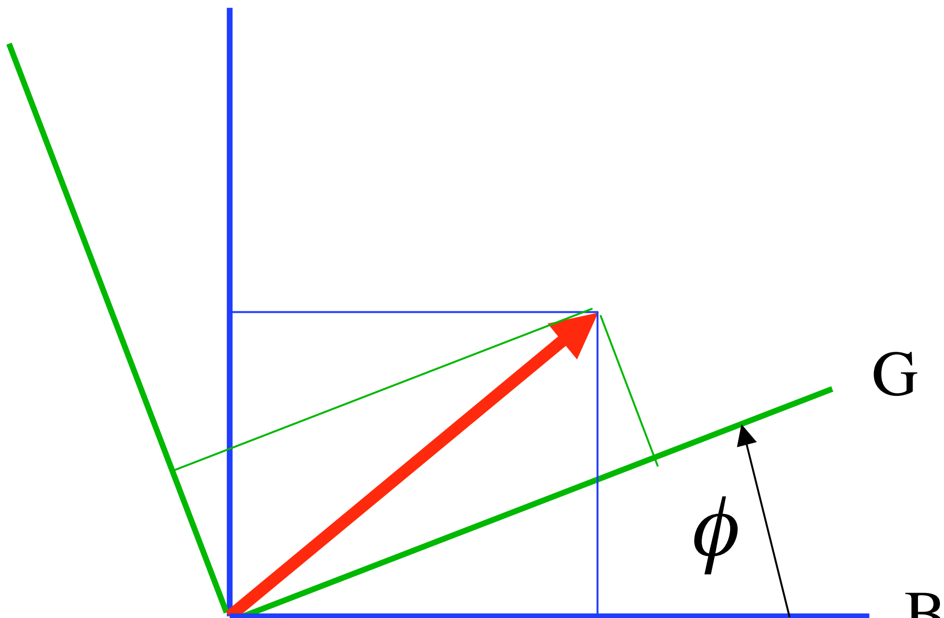
$$\mathbf{x}_{AC} = \begin{bmatrix} x_{AC} & y_{AC} & \phi_{AC} \end{bmatrix}^T$$

$$\mathbf{x}_{BC} = \begin{bmatrix} x_{BC} & y_{BC} & \phi_{BC} \end{bmatrix}^T$$

$$\mathbf{x}_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T$$

Rotation Matrix

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_G \\ y_G \end{bmatrix} = \begin{bmatrix} x_B \\ y_B \end{bmatrix}$$



Compounding

- Let $\mathbf{x}_{AB}$ be the pose of object B in the frame of reference of A. (Sometimes written B_A.)
- Given $\mathbf{x}_{AB}$ and $\mathbf{x}_{BC}$, calculate $\mathbf{x}_{AC}$.

$$\mathbf{x}_{AC} = \mathbf{x}_{AB} \oplus \mathbf{x}_{BC} = \begin{bmatrix} x_{BC}\cos\phi_{AB} - y_{BC}\sin\phi_{AB} + x_{AB} \\ x_{BC}\sin\phi_{AB} + y_{BC}\cos\phi_{AB} + y_{AB} \\ \phi_{AB} + \phi_{BC} \end{bmatrix}$$

- Compute $\mathbf{C}(\mathbf{x}_{AC})$ from $\mathbf{C}(\mathbf{x}_{AB})$, $\mathbf{C}(\mathbf{x}_{BC})$, and $\mathbf{C}(\mathbf{x}_{AB,}\mathbf{x}_{BC})$.

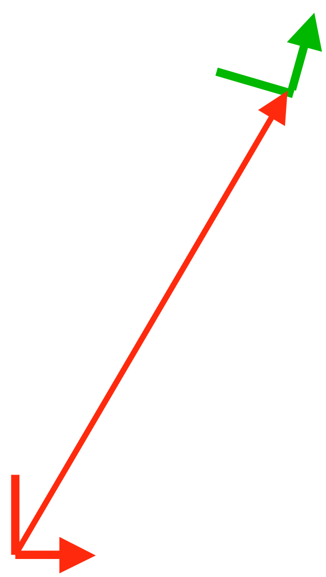
Computing Covariance

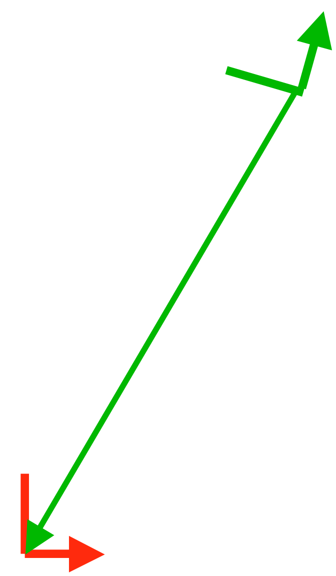
- Consider the linear mapping **y = Mx+b**

$$\begin{aligned}
\mathbf{y} &= \mathbf{M}\mathbf{x} + \mathbf{b} \\
\mathbf{C}(\mathbf{y}) &= \mathbf{C}(\mathbf{M}\mathbf{x} + \mathbf{b}) \\
&= E[(\mathbf{M}\mathbf{x} + \mathbf{b} - (\mathbf{M}\hat{\mathbf{x}} + \mathbf{b}))\ (\mathbf{M}\mathbf{x} + \mathbf{b} - (\mathbf{M}\hat{\mathbf{x}} + \mathbf{b}))^{\mathrm{T}}] \\
&= E[\mathbf{M}(\mathbf{x} - \hat{\mathbf{x}})\ (\mathbf{M}(\mathbf{x} - \hat{\mathbf{x}}))^{\mathrm{T}}] \\
&= E[\mathbf{M}\ (\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}] \\
&= \mathbf{M}\,E[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^{\mathrm{T}}]\mathbf{M}^{\mathrm{T}} \\
&= \mathbf{M}\,\mathbf{C}(\mathbf{x})\,\mathbf{M}^{\mathrm{T}}
\end{aligned}$$

- Apply this to nonlinear functions by using Taylor Series.

Inverse Relationship





$$\mathbf{x}_{AB} = \begin{bmatrix} x_{AB} & y_{AB} & \phi_{AB} \end{bmatrix}^T$$

$$\mathbf{x}_{BA} = \begin{bmatrix} x_{BA} & y_{BA} & \phi_{BA} \end{bmatrix}^T$$

The Inverse Relationship

- Let $\mathbf{x}_{AB}$ be the pose of object B in the frame of reference of A.
- Given $\mathbf{x}_{AB}$, calculate $\mathbf{x}_{BA}$.

$$\mathbf{x}_{BA} = (-)\mathbf{x}_{AB} = \begin{bmatrix} -x_{AB}\cos\phi_{AB} - y_{AB}\sin\phi_{AB} \\ x_{AB}\sin\phi_{AB} - y_{AB}\cos\phi_{AB} \\ -\phi_{AB} \end{bmatrix}$$

- Compute $\mathbf{C}(\mathbf{x}_{BA})$ from $\mathbf{C}(\mathbf{x}_{AB})$

Composite Relationships

- Compounding combines relationships head-to-tail: $\mathbf{x}_{AC} = \mathbf{x}_{AB} \oplus \mathbf{x}_{BC}$
- Tail-to-tail combinations come from observing two things from the same point: $\mathbf{x}_{BC} = (\ominus \mathbf{x}_{AB}) \oplus \mathbf{x}_{AC}$
- Head-to-head combinations come from two observations of the same thing: $\mathbf{x}_{AC} = \mathbf{x}_{AB} \oplus (\ominus \mathbf{x}_{CB})$
- They provide new relationships between their endpoints.

Merging Information

- An uncertain observation of a pose is combined with previous knowledge using the extended Kalman filter.
  - Previous knowledge: $\mathbf{x}_k^-, \quad \mathbf{P}_k^-$
  - New observation: $\mathbf{z}_k, \quad \mathbf{R}$

- Update: $\mathbf{x} = \mathbf{x}(\text{new}) \otimes \mathbf{x}(\text{old})$

- Can integrate dynamics as well.

EKF Update Equations

- Predictor step: $\hat{\mathbf{x}}_k^- = f(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_k)$

$$\mathbf{P}_k^- = \mathbf{A}\mathbf{P}_{k-1}\mathbf{A}^T + \mathbf{Q}$$

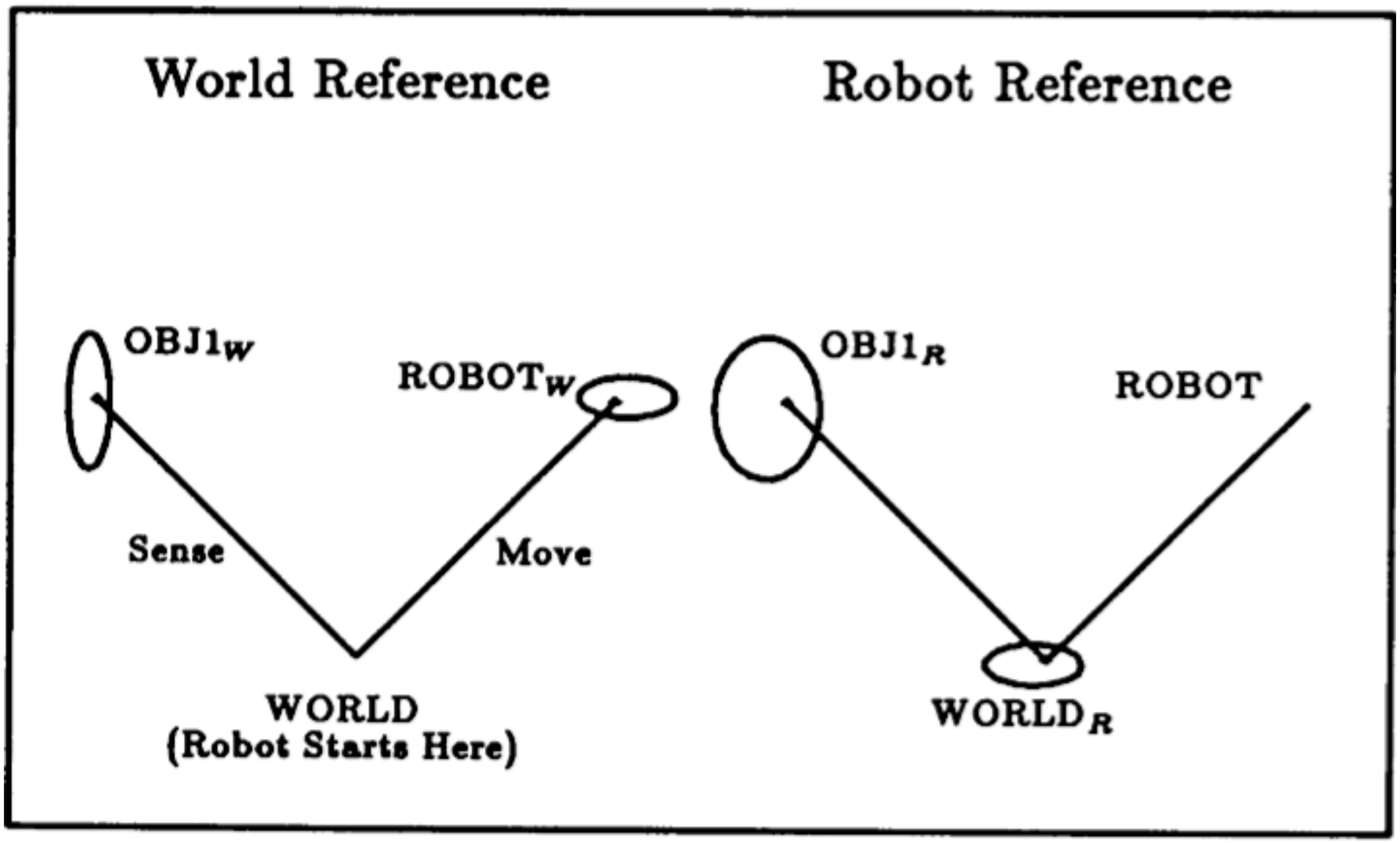
- Kalman gain: $\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}^T (\mathbf{H}\mathbf{P}_k^- \mathbf{H}^T + \mathbf{R})^{-1}$

- Corrector step: $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{z}_k - h(\hat{\mathbf{x}}_k^-))$

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_k^-$$

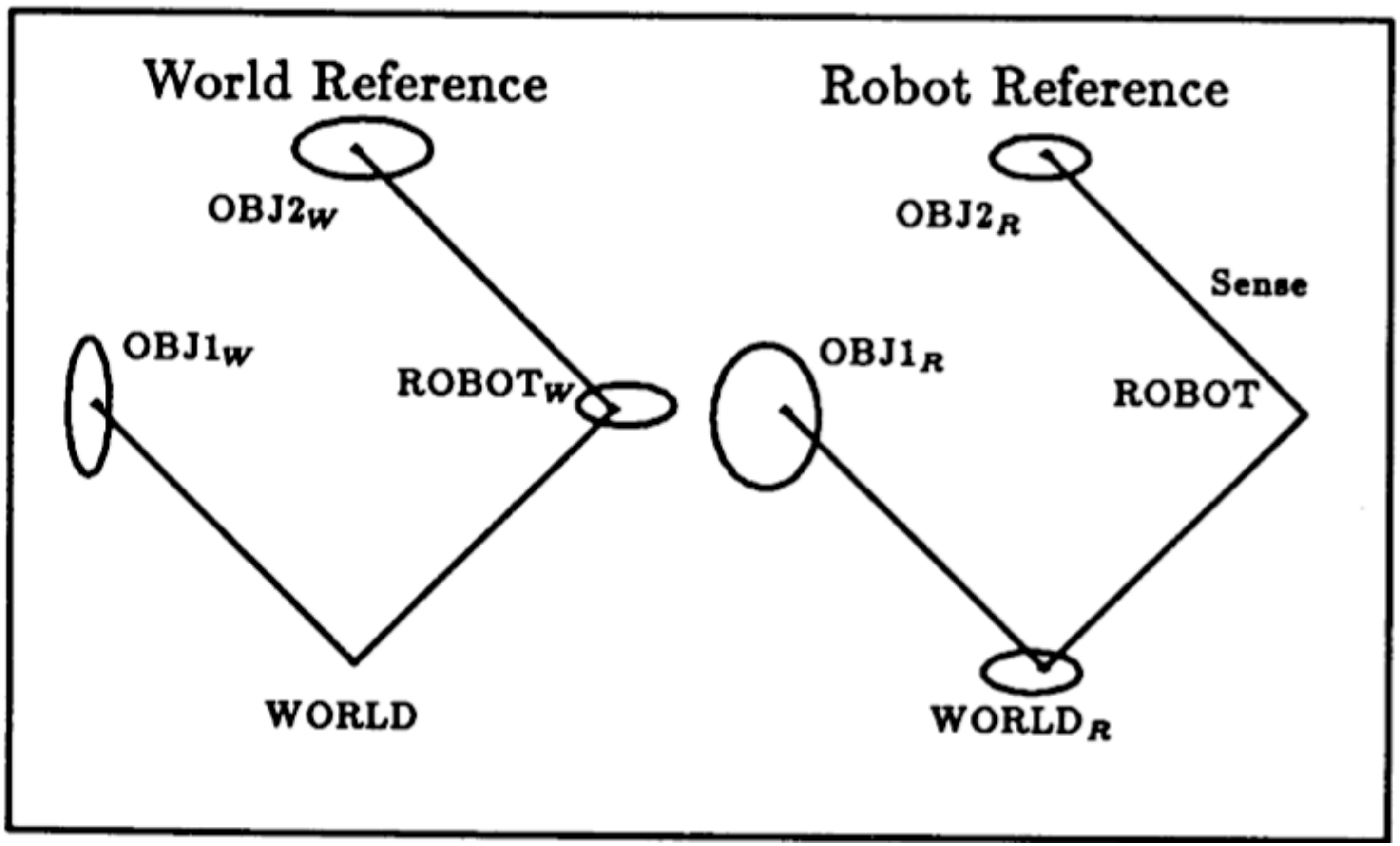
Reversing and Compounding

- $OBJ1_R = (\ominus ROBOT_W) \oplus OBJ1_W$
- $= WORLD_R \oplus OBJ1_W$



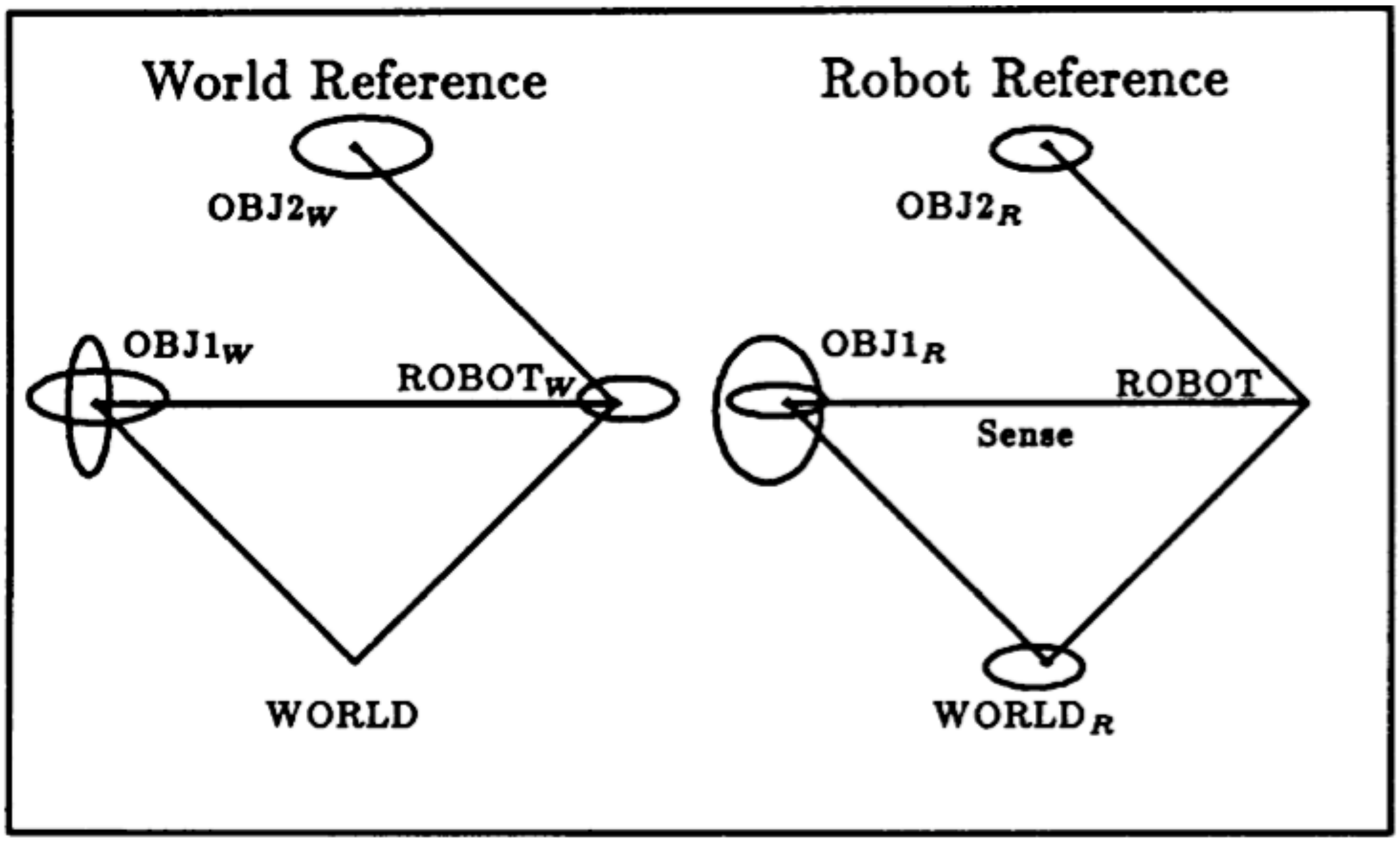
Sensing Object #2

- $OBJ2_W = ROBOT_W \oplus OBJ2_R$



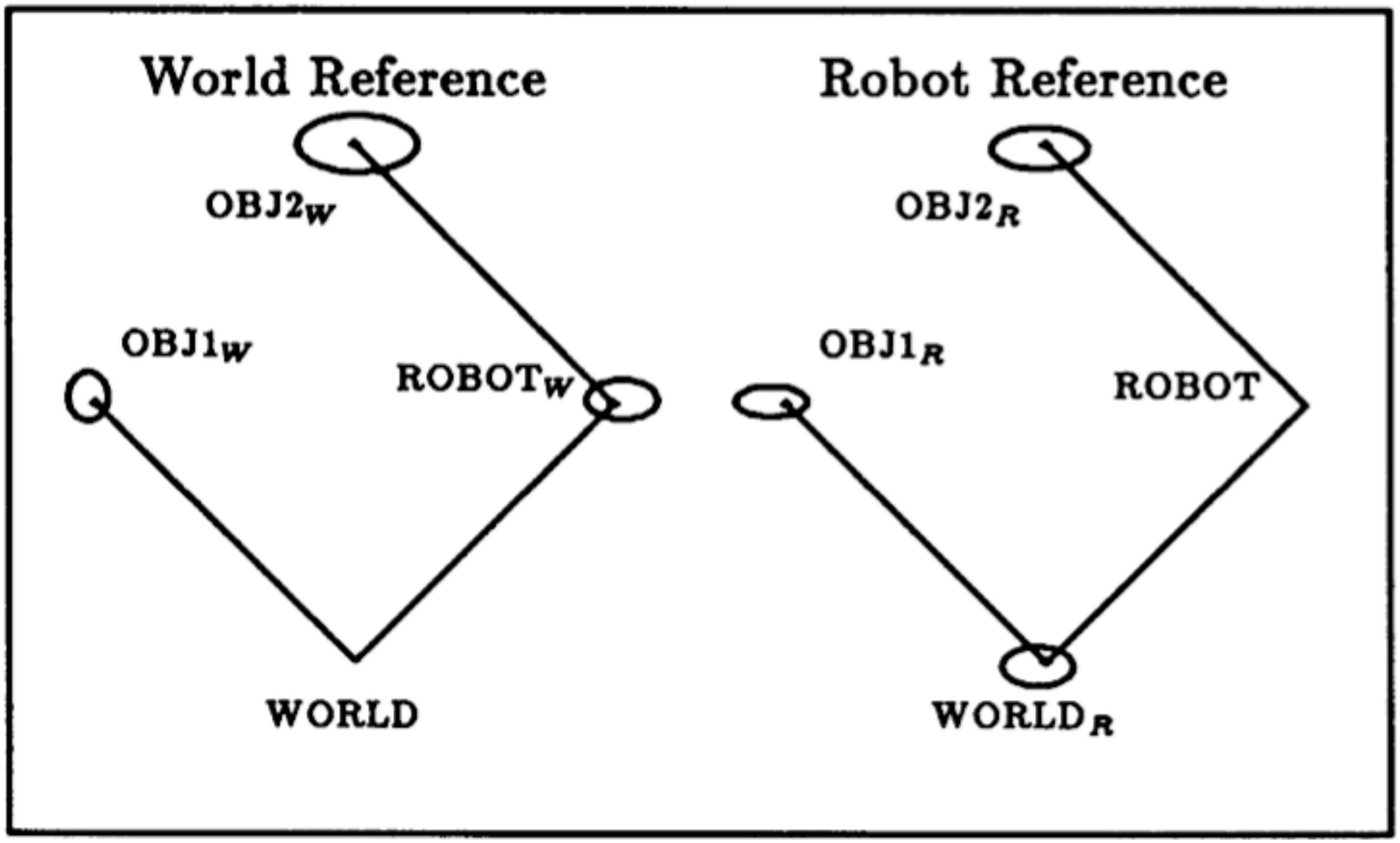
Observing Object #1 Again

- $OBJ1_W = ROBOT_W \oplus OBJ1_R$



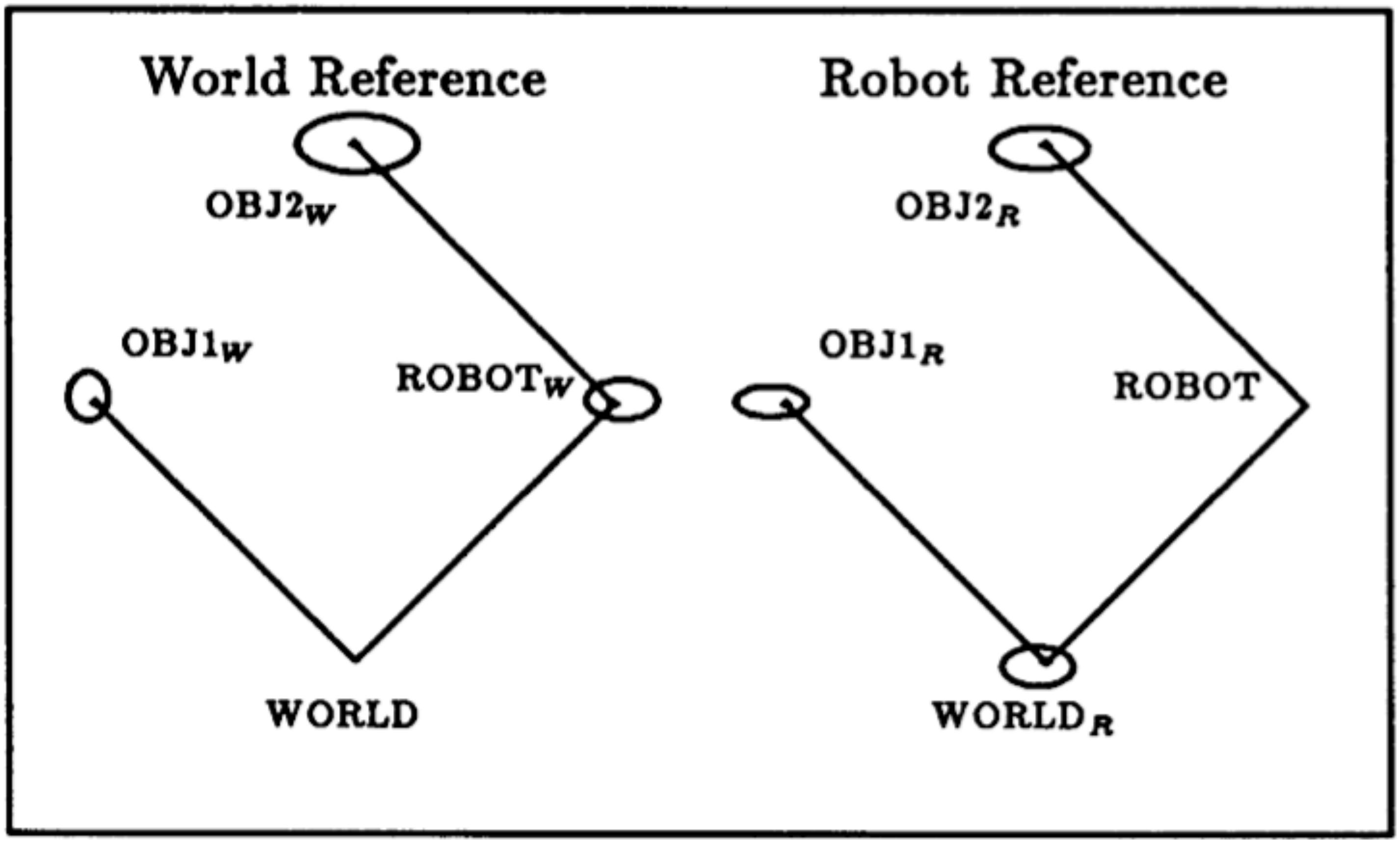
Combining Observations (1)

- $OJB1_W = OJB1_W(new) \otimes OBJ1_W(old)$
- $OJB1_R = OJB1_R(new) \otimes OBJ1_R(old)$



Combining Observations (2)

- $ROBOT_W(new) = OJB1_W \oplus (\ominus OBJ1_R)$
- $ROBOT_W = ROBOT_W(new) \otimes ROBOT_W(old)$



Useful for Feature-Based Maps

- We'll see this again when we study FastSLAM.