Graphs
Topic 21

"Hopefully, you've played around a bit with The Oracle of Bacon at Virginia and discovered how few steps are necessary to link just about anybody who has ever been in a movie to Kevin Bacon, but could there be some actor or actress who is even closer to the center of the Hollywood universe?.

By processing all of the almost half of a million people in the Internet Movie Database I discovered that there are currently 1160 people who are better centers than Kevin Bacon! ... By computing the average of these numbers we see that the average (Sean) Connery Number is about 2,682 making Connery a better center than Bacon"

-Who is the Center of the Hollywood Universe?, University of Virginia
That was in 2001.
In 2013 Harvey Keitel has become the center of the Hollywood Universe. Connery is 136th. Bacon has moved up to 370th.

An Early Problem in
Graph Theory

- Leonhard Euler (1707 - 1783)
  – One of the first mathematicians to study graphs
- The Seven Bridges of Konigsberg Problem
- A puzzle for the residents of the city
- The river Pregel flows through the city
- 7 bridges cross the river
- Can you cross all bridges while crossing each bridge only once?

Konigsberg and the River Pregel

Clicker Question 1

- How many solutions does the Seven Bridges of Konigsberg Problem have?
  A. 0
  B. 1
  C. 2
  D. 3
  E. 4 or more
How to Solve

- Brute Force?
- Euler's Solution
  - Redraw the map as a graph (really a multigraph)

Graph Definitions

- A graph is comprised of a set of vertices (nodes) and a set of edges (links, arcs) connecting the vertices
- In a directed graph edges are one-way
  - movement allowed from first node to second, but not second to first
  - directed graphs also called digraphs
- In an undirected graph edges are two-way
  - movement allowed in either direction

Euler's Proposal

- A connected graph has an Euler tour (possible to cross each edge only once while traversing each edge only once and returning to starting point) if and only if every vertex has an even number of edges
  - Eulerian Circuit
- What if we reduce the problem to only crossing each exactly once edge once?
  - Doesn't matter if we end up where we started
  - Eulerian Trail

Definitions

- In a weighted graph the edge has cost or weight that measures the cost of traveling along the edge
- A path is a sequence of vertices connected by edges
  - The path length is the number of edges
  - The weighted path length is the sum of the cost of the edges in a path
- A cycle is a path of length 1 or more that starts and ends at the same vertex
  - a directed acyclic graph is a directed graph with no cycles
Graphs We've Seen

Example Graph

- Computer Scientists use graphs to model all kinds of things

Arpanet 1969, 1971

Example Graph

Enron emails 2001

Example Graph

Facebook Graph API
Example Graph

Adjacency Matrix Representation

Representing Graphs

The Map Coloring Problem
A Solution

What About the Ocean?

<table>
<thead>
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Make the Ocean Blue

More Definitions

- **A dense graph** is one with a large number of edges
  - maximum number of edges?
- **A sparse graph** is one in which the number of edges is much less than the maximum possible number of edges
  - No standard cutoff for dense and sparse graphs
Graph Representation

- For dense graphs the adjacency matrix is a reasonable choice
  - For weighted graphs change booleans to cost
  - Can the adjacency matrix handle directed graphs?
- Most graphs are sparse, not dense
- For sparse graphs an adjacency list is an alternative that uses less space
- Each vertex keeps a list of vertices it is connected to.

Graph Implementation

```java
public class Graph
{
    private static final double INFINITY = Double.MAX_VALUE;
    private Map<String, Vertex> vertices;

    public Graph() // create empty Graph
    {
        // find all paths from given vertex
        public void findUnweightedShortestPath(String startName)
        {
            // called after findUnweightedShortestPath
            public void printPath(String destName)
            {
            }
        }
    }
}
```

Graph Class

- This Graph class stores vertices
- Each vertex has an adjacency list
  - what vertices does it connect to?
- shortest path method finds all paths from start vertex to every other vertex in graph
- after shortest path method called queries can be made for path length from start node to destination node

Vertex Class (nested in Graph)

```java
private static class Vertex
{
    private String name;
    private List<Edge> adjacent;

    public Vertex(String n)
    {
        // for shortest path algorithms
        private double distance;
        private Vertex prev;
        private int scratch;

        // call before finding new paths
        public void reset()
        {
        }
    }
}```
Edge Class (nested in Graph)

private static class Edge  
private Vertex dest;  
private double cost;  

private Edge(Vertex d, double c)

Word Ladders

- Agree upon dictionary
- Start word and end word of same length
- Change one letter at a time to form step
- Step must also be a word
- Example: Start = silly, end = funny

Unweighted Shortest Path

- Given a vertex, S (for start) find the shortest path from S to all other vertices in the graph
- Graph is unweighted (set all edge costs to 1)

Graph Representation

- What are the vertices and when does an edge exist between two vertices?

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Edges</th>
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<tbody>
<tr>
<td>A. Letters</td>
<td>Words</td>
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<td>B. Words</td>
<td>Words that share one or more letters</td>
</tr>
<tr>
<td>C. Letters</td>
<td>Words that share one or more letters</td>
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<tr>
<td>D. Words</td>
<td>Words that differ by one letter</td>
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<tr>
<td>E. Words</td>
<td>Letters</td>
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</tbody>
</table>
Unweighted Shortest Path Algorithm

- Problem: Find the shortest word ladder between two words if one exists
- What kind of search should we use?

A. Breadth First Search
B. Depth First Search
C. Either one

Size of Graph

- Number of vertices and edges depends on dictionary
- Modified Scrabble dictionary, 5 letter words
- Words are vertices
  - 8660 words
- Edge exists between word if they are one letter different
  - 24,942 edges

Is this graph sparse or dense?
A. Sparse
B. Dense

Max number of edges = $N \times (N - 1) / 2$

37,493,470
Portion of Graph

Start at "smart" and enqueue it [smart]

Dequeue (smart), loop through edges [swart]

Dequeue (smart), loop through edges [swart, start]
Dequeue (smart), loop through edges [swart, start, scar]  

Dequeue (smart), loop through edges [swart, start, scar, smalt]  

Dequeue (smart), loop through edges [swart, start, scar, smalt, smarm]  

Done with smart, dequeue (swart) [start, scar, smalt, smarm]
loop through edges of swath (start already present) [start, scart, smalt, smarm]

loop through edges of swath (scart already present) [start, scart, smalt, smarm]

loop through edges of swath [start, scart, smalt, smarm, swarm]

loop through edges of swath [start, scart, smalt, smarm, swarm, sware]
Unweighted Shortest Path

- Implement method
demo
how is path printed?
The diameter of a graph is the longest shortest past in the graph
How to find?
How to find center of graph?
- vertex connected to the largest number of other vertices with the shortest average path length

Positive Weighted Shortest Path

- Edges in graph are weighted and all weights are positive
- Similar solution to unweighted shortest path
- Dijkstra's algorithm
- Edsger W. Dijkstra (1930–2002)
- UT Professor 1984 - 2000
- Algorithm developed in 1956 and published in 1959.

Dijkstra's Algorithm

- Pick the start vertex
- Set the cost of the start vertex to 0 and all other vertices to INFINITY
- While there are unvisited vertices:
  - Let the current vertex be the lowest cost vertex that has not yet been visited
  - mark current vertex as visited
  - for each edge from the current vertex
    - if the sum of the cost of the current vertex and the cost of the edge is less than the cost of the destination vertex
      - update the cost of the destination vertex
      - set the previous of the destination vertex to the current vertex

Example of a Greedy Algorithm
- A Greedy Algorithm does what appears to be the best thing at each stage of solving a problem
- Gives best solution in Dijkstra's Algorithm
- Does NOT always lead to best answer
- Fair teams:
  - (10, 10, 8, 8, 8), 2 teams
- Making change with fewest coins
  - 1, 5, 10 cents
  - 1, 5, 12 cents
What is the lowest cost path from A to E?
A. 3
B. 17
C. 20
D. 28
E. 37

A is start vertex
Set cost of A to 0, all others to INFINITY
Place A in a priority queue

[(A,0)] \text{pq}
dequeue (A,0)
Mark A as visited

[current vertex A:] loop through A’s edges
if sum of cost from A to edge is less than current cost
update cost and prev
[ ] pq
A -> B, 0 + 1 < INFINITY
[(B, 1)] pq

[(B, 1)] pq
A -> C, 0 + 7 < INFINITY
[(B, 1), (C, 7)] pq

[(B, 1), (C, 7)] pq
A -> G, 0 + 17 < INFINITY
[(B, 1), (C, 7), (G, 17)] pq

[(B, 1), (C, 7), (G, 17)] pq
current vertex B:
loop through B’s edges
if sum of cost from B to edge is less than current cost
update cost and prev
[(C, 7), (G, 17)] pq
B -> C, 1 + 3 < 7
update C's cost and previous
[(C, 4), (C, 7), (G, 17)] pq

[(C, 4), (C, 7), (G, 17)] pq
B -> D, 1 + 21 < INFINITY
[(C, 4), (C, 7), (G, 17), (D, 22)] pq

[(C, 4), (C, 7), (G, 17), (D, 22)] pq
current vertex is C, cost 4
loop through C's edges

[(C, 7), (G, 17), (D, 22)] pq
C -> A, A already visited so skip
[(C, 7), (G, 17), (D, 22)] pq
C -> B, B already visited so skip

[(C, 7), (G, 17), (D, 22)] pq
C -> F, 4 + 3 < INFINITY
[(C, 7), (F, 7), (G, 17), (D, 22)] pq

[(C, 7), (F, 7), (G, 17), (D, 22)] pq
current vertex is C
Already visited so skip

[(F, 7), (G, 17), (D, 22)] pq
current vertex is F
loop through F's edges
[(G, 17), (D, 22)] pq
F -> C, already visited so skip

[(G, 17), (D, 22)] pq
F -> D, 7 + 4 < 22
update D's cost and previous
[(D, 11), (G, 17), (D, 22)] pq

[(D, 11), (G, 17), (D, 22)] pq
current vertex is D
loop through D's edges

[(G, 17), (D, 22)] pq
D -> B, already visited so skip
[(G, 17), (D, 22)] pq
D -> E, 11 + 6 < INFINITY
update E's cost and previous
[(G, 17), (E, 17), (D, 22)] pq

[(G, 17), (E, 17), (D, 22)] pq
D -> F, already visited so skip

[(G, 17), (E, 17), (D, 22)] pq
D -> G, 11 + 5 < 17
update G's cost and previous
[(G, 16), (G, 17), (E, 17), (D, 22)] pq

[(G, 17), (E, 17), (D, 22)] pq
current vertex is G
loop though edges, already visited all neighbors
Implementing Dijkstra's

- Create a Path class to allow for multiple distances to a given vertex

```java
private static class Path
    implements Comparable<Path> {
    private Vertex dest;
    private double cost;

    // Use a priority queue of Paths to store the vertices and distances
```

Alternatives to Dijkstra's Algorithm

- A*, pronounced "A Star"
- A heuristic, goal of finding shortest weighted path from single start vertex to goal vertex
- Uses actual distance like Dijkstra's but also estimates remaining cost or distance
  - distance is set to current distance from start PLUS the estimated distance to the goal
- For example when finding a path between towns, estimate the remaining distance as the straight-line (as the crow files) distance between current location and goal.
Spanning Tree

- **Spanning Tree**: A tree of edges that connects all the vertices in a graph

```
A  1  17  5  B
   ↑    ↑    ↓
  C    D   F
   7  3  21  4
```

Minimum Spanning Tree

- **Minimum Spanning Tree**: A spanning tree in a weighted graph with the lowest total cost
- used in network design, taxonomy, image registration, and more!

Cost of spanning tree shown?
A. 6
B. 7
C. 29
D. 61
E. None of These

Prim's Algorithm

- Pick a vertex arbitrarily from graph
  - In other words, it doesn't matter which one
- Add lowest cost edge between the tree and a vertex that is not part of the graph UNTIL every vertex is part of the tree
- Greedy Algorithm, very similar to Dijkstra's

```
Pick D as root
```

```
A  4  2  5  C
   ↑  3  8  F
  B    ↑  7  G
   2  4  6  1
```

```
A  2  3  11  B
   ↑  4  7  E
  C    ↑  5  D
   2  8  6  F
```
Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
2 from D to A (or C)

CS314  Graphs  81

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
2 from D to C (OR from A or B)

CS314  Graphs  82

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
2 from A to B

CS314  Graphs  83

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
5 from D to G

CS314  Graphs  84
Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
1 from G to F

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
6 from G to E

Pick D as root

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
4 from D to F
Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
3 from F to C

CS314  Graphs  89

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
3 from C to B

CS314  Graphs  90

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
1 from B to A

CS314  Graphs  91

Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
5 from D to G

CS314  Graphs  92
Prim's Algorithm

Lowest cost edge from tree to vertex not in Tree?
6 from D to E

Cost of Spanning Tree?

Other Graph Algorithms

- Lots!
- [http://en.wikipedia.org/wiki/Category:Graph_algorithms](http://en.wikipedia.org/wiki/Category:Graph_algorithms)