"bit twiddling: 1. (pejorative) An exercise in tuning (see tune) in which incredible amounts of time and effort go to produce little noticeable improvement, often with the result that the code becomes incomprehensible."

- The Hackers Dictionary, version 4.4.7
Clicker Question 1

"My program finds all the primes between 2 and 1,000,000,000 in 1.37 seconds."

– how efficient is my solution, in terms of time?

A. Good
B. Bad
C. It depends
Efficiency

- Computer Scientists don’t just write programs.
- They also *analyze* them.
- How efficient is a program?
  - How much time does it take program to complete?
  - How much memory does a program use?
  - How do these change as the amount of data changes?
  - What is the difference between the best case and worst case efficiency if any?
Technique

- Informal approach for this class
  - more formal techniques in theory classes

- Many simplifications
  - view algorithms as Java programs
  - count executable statements in program or method
  - find number of statements as function of the amount of data
  - focus on the dominant term in the function
Counting Statements

```java
int x; // one statement
x = 12; // one statement
int y = z * x + 3 % 5 * x / i; // 1
x++; // one statement
boolean p = x < y && y % 2 == 0 ||
    z >= y * x; // 1
int[] list = new int[100]; // 100
list[0] = x * x + y * y; // 1
```
Clicker Question 2

What is output by the following code?

```java
int total = 0;
for(int i = 0; i < 13; i++)
    for(int j = 0; j < 11; j++)
        total += 2;
System.out.println(total);
```

A. 24  
B. 120  
C. 143  
D. 286  
E. 338
Clicker Question 3

What is output when method `sample` is called?

```java
// pre: n >= 0, m >= 0
public static void sample(int n, int m) {
    int total = 0;
    for(int i = 0; i < n; i++)
        for(int j = 0; j < m; j++)
            total += 5;
    System.out.println( total );
}
```

A. 5  D. $n^m$
B. $n \times m$  E. $(n \times m)^5$
C. $n \times m \times 5$
Example

```java
public int total(int[] values) {
    int result = 0;
    for(int i = 0; i < values.length; i++)
        result += values[i];
    return result;
}
```

- How many statements are executed by method `total` as a function of `values.length`?
- Let N = `values.length`
  - N is commonly used as a variable that denotes the amount of data.
Counting Up Statements

- `int result = 0;` \( N + 1 \)
- `int i = 0;` \( N + 1 \)
- `i < values.length;` \( N + 1 \)
- `i++` \( N \)
- `result += values[i];` \( N \)
- `return total;` \( N + 1 \)

- \( T(N) = 3N + 4 \)
- \( T(N) \) is the number of executable statements in method `total` as function of `values.length`
Another Simplification

- When determining complexity of an algorithm we want to simplify things
  - hide some details to make comparisons easier
- Like assigning your grade for course
  - At the end of CS314 your transcript won’t list all the details of your performance in the course
  - it won’t list scores on all assignments, quizzes, and tests
  - simply a letter grade, B- or A or D+
- So we focus on the dominant term from the function and ignore the coefficient
Big O

- The most common method and notation for discussing the execution time of algorithms is *Big O*, also spoken *Order*
- Big O is the *asymptotic execution time* of the algorithm
- Big O is an upper bound
- It is a mathematical tool
- Hide a lot of unimportant details by assigning a simple grade (function) to algorithms
Formal Definition of Big O

- $T(N)$ is $O(F(N))$ if there are positive constants $c$ and $N_0$ such that $T(N) \leq cF(N)$ when $N \geq N_0$
  - $N$ is the size of the data set the algorithm works on
  - $T(N)$ is a function that characterizes the actual running time of the algorithm
  - $F(N)$ is a function that characterizes an upper bounds on $T(N)$. It is a limit on the running time of the algorithm. (The typical Big functions table)
  - $c$ and $N_0$ are constants
What it Means

- $T(N)$ is the actual growth rate of the algorithm
  - can be equated to the number of executable statements in a program or chunk of code
- $F(N)$ is the function that bounds the growth rate
  - may be upper or lower bound
- $T(N)$ may not necessarily equal $F(N)$
  - constants and lesser terms ignored because it is a bounding function
Showing $O(N)$ is Correct

- Recall the formal definition of Big O
  - $T(N)$ is $O(F(N))$ if there are positive constants $c$ and $N_0$ such that $T(N) \leq cF(N)$ when $N > N_0$

- Recall method $total$, $T(N) = 3N + 4$
  - show method $total$ is $O(N)$.
  - $F(N)$ is $N$

- We need to choose constants $c$ and $N_0$

- how about $c = 4$, $N_0 = 5$?
vertical axis: time for algorithm to complete. (simplified to number of executable statements)

c * F(N), in this case, c = 4, c * F(N) = 4N

T(N), actual function of time. In this case 3N + 4

F(N), approximate function of time. In this case N

N₀ = 5

horizontal axis: N, number of elements in data set
## Typical Big O Functions – "Grades"

<table>
<thead>
<tr>
<th>Function</th>
<th>Common Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>N!</td>
<td>factorial</td>
</tr>
<tr>
<td>$2^N$</td>
<td>Exponential</td>
</tr>
<tr>
<td>$N^d$, $d &gt; 3$</td>
<td>Polynomial</td>
</tr>
<tr>
<td>$N^3$</td>
<td>Cubic</td>
</tr>
<tr>
<td>$N^2$</td>
<td>Quadratic</td>
</tr>
<tr>
<td>$N \sqrt[2]{N}$</td>
<td>N Square root N</td>
</tr>
<tr>
<td>$N \log N$</td>
<td>N log N</td>
</tr>
<tr>
<td>$N$</td>
<td>Linear</td>
</tr>
<tr>
<td>$\sqrt{N}$</td>
<td>Root - n</td>
</tr>
<tr>
<td>$\log N$</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>1</td>
<td>Constant</td>
</tr>
</tbody>
</table>
Clicker Question 4

Which of the following is true?

A. Method $\text{total}$ is $O(N^{1/2})$
B. Method $\text{total}$ is $O(N)$
C. Method $\text{total}$ is $O(N^2)$
D. Two of A – C are correct
E. All of three of A – C are correct
Dealing with other methods

- What do I do about method calls?
  
  ```java
  double sum = 0.0;
  for(int i = 0; i < n; i++)
      sum += Math.sqrt(i);
  ```

- Long way
  - go to that method or constructor and count statements

- Short way
  - substitute the simplified Big O function for that method.
  - if Math.sqrt is constant time, O(1), simply count
    ```java
    sum += Math.sqrt(i);
    ```
    as one statement.
Dealing With Other Methods

public int foo(int[] list) {
    int total = 0;
    for(int i = 0; i < list.length; i++)
        total += countDups(list[i], list);
    return total;
}

// method countDups is O(N) where N is the
// length of the array it is passed

What is the Big O of foo?
A. O(1)     B. O(N)     C. O(NlogN)
D. O(N^2)   E. O(N!)
Independent Loops

// from the Matrix class
public void scale(int factor) {
    for(int r = 0; r < numRows(); r++)
        for(int c = 0; c < numCols(); c++)
            iCells[r][c] *= factor;
}

Assume an numRows() = N and numCols() = N.
In other words, a square Matrix.
numRows and numCols are O(1)

What is the T(N)? What is the Big O?
A. O(1)  B. O(N)  C. O(NlogN)
D. O(N^2)  E. O(N!)

CS 314  Efficiency - Complexity  20
Just Count Loops, Right?

// assume mat is a 2d array of booleans
// assume mat is square with N rows,
// and N columns

int numThings = 0;
for (int r = row - 1; r <= row + 1; r++)
    for (int c = col - 1; c <= col + 1; c++)
        if (mat[r][c])
            numThings++;

What is the order of the above code?
A. O(1)   B. O(N)   C. O(N^2)   D. O(N^3)   E. O(N^{1/2})
It is Not Just Counting Loops

// Second example from previous slide could be rewritten as follows:
int numThings = 0;
if( mat[r-1][c-1] ) numThings++;
if( mat[r-1][c] ) numThings++;
if( mat[r-1][c+1] ) numThings++;
if( mat[r][c-1] ) numThings++;
if( mat[r][c] ) numThings++;
if( mat[r][c+1] ) numThings++;
if( mat[r+1][c-1] ) numThings++;
if( mat[r+1][c] ) numThings++;
if( mat[r+1][c+1] ) numThings++;
Sidetrack, the logarithm

- Thanks to Dr. Math
- $3^2 = 9$
- Likewise $\log_3 9 = 2$
  - "The log to the base 3 of 9 is 2."
- The way to think about log is:
  - "The log to the base x of y is the number you can raise x to to get y."
  - Say to yourself "The log is the exponent." (and say it over and over until you believe it.)
  - In CS we work with base 2 logs, a lot
- $\log_2 32 = ? \quad \log_2 8 = ? \quad \log_2 1024 = ? \quad \log_{10} 1000 = ?$
When Do Logarithms Occur

- Algorithms tend to have a logarithmic term when they use a divide and conquer technique
- the data set keeps getting divided by 2

```java
public int foo(int n) {
    // pre n > 0
    int total = 0;
    while( n > 0 ) {
        n = n / 2;
        total++;
    }
    return total;
}
```

What is the order of the above code?

A. O(1)  
B. O(logN)  
C. O(N)  
D. O(Nlog N)  
E. O(N^2)
Significant Improvement – Algorithm with Smaller Big O function

Problem: Given an array of ints replace any element equal to 0 with the maximum positive value to the right of that element. (if no positive value to the right, leave unchanged.)

Given:

\[0, 9, 0, 13, 0, 0, 7, 1, -1, 0, 1, 0]\]

Becomes:

\[13, 9, 13, 13, 7, 7, 7, 1, -1, 1, 1, 0]\]
Replace Zeros – Typical Solution

```java
public void replace0s(int[] data) {
    for (int i = 0; i < data.length - 1; i++) {
        if (data[i] == 0) {
            int max = 0;
            for (int j = i + 1; j < data.length; j++)
                max = Math.max(max, data[j]);
            data[i] = max;
        }
    }
}
```

Assume all values are zeros. (worst case)
Example of a **dependent loops**.
Replace Zeros – Alternate Solution

```java
public void replace0s(int[] data){
    int max =
        Math.max(0, data[data.length – 1]);
    int start = data.length – 2;
    for(int i = start; i >= 0; i--){
        if( data[i] == 0 )
            data[i] = max;
        else
            max = Math.max(max, data[i]);
    }
}

Big O of this approach?
A. O(1)   B. O(N)   C. O(NlogN)
D. O(N^2)  E. O(N!)
```
A Useful Proportion

- Since \( F(N) \) is characterizes the running time of an algorithm the following proportion should hold true:

\[
\frac{F(N_0)}{F(N_1)} \approx \frac{\text{time}_0}{\text{time}_1}
\]

- An algorithm that is \( O(N^2) \) takes 3 seconds to run given 10,000 pieces of data.
  - How long do you expect it to take when there are 30,000 pieces of data?
  - common mistake
  - logarithms?
Why Use Big O?

- As we build data structures Big O is the tool we will use to decide under what conditions one data structure is better than another.
- Think about performance when there is a lot of data.
  - "It worked so well with small data sets..."
  - Joel Spolsky, Schlemiel the painter's Algorithm
- Lots of trade offs
  - some data structures good for certain types of problems, bad for other types
  - often able to trade SPACE for TIME.
  - Faster solution that uses more space
  - Slower solution that uses less space
Big O Space

- Big O could be used to specify how much space is needed for a particular algorithm – in other words how many variables are needed
- Often there is a *time – space tradeoff* – can often take less time if willing to use more memory – can often use less memory if willing to take longer – truly beautiful solutions take less time and space

*The biggest difference between time and space is that you can't reuse time.* - Merrick Furst
Quantifiers on Big O

- It is often useful to discuss different cases for an algorithm
- Best Case: what is the best we can hope for?
  - least interesting
- Average Case (a.k.a. expected running time): what usually happens with the algorithm?
- Worst Case: what is the worst we can expect of the algorithm?
  - very interesting to compare this to the average case
Best, Average, Worst Case

- To Determine the best, average, and worst case Big O we must make assumptions about the data set
  - Best case -> what are the properties of the data set that will lead to the fewest number of executable statements (steps in the algorithm)
  - Worst case -> what are the properties of the data set that will lead to the largest number of executable statements
  - Average case -> Usually this means assuming the data is randomly distributed
    - or if I ran the algorithm a large number of times with different sets of data what would the average amount of work be for those runs?
public double minimum(double[] values) {
    int n = values.length;
    double minValue = values[0];
    for (int i = 1; i < n; i++)
        if (values[i] < minValue)
            minValue = values[i];
    return minValue;
}

- T(N)? F(N)? Big O? Best case? Worst Case?
  Average Case?
- If no other information, assume asking average case
Example of Dominance

- Look at an extreme example. Assume the actual number as a function of the amount of data is:

\[ \frac{N^2}{10000} + 2N \log_{10} N + 100000 \]

- Is it plausible to say the \( N^2 \) term dominates even though it is divided by 10000 and that the algorithm is \( O(N^2) \)?

- What if we separate the equation into \( (\frac{N^2}{10000}) \) and \( (2N \log_{10} N + 100000) \) and graph the results.
For large values of $N$ the $N^2$ term dominates so the algorithm is $O(N^2)$

When does it make sense to use a computer?
Comparing Grades

- Assume we have a problem
- Algorithm A solves the problem correctly and is $O(N^2)$
- Algorithm B solves the same problem correctly and is $O(N \log_2 N)$
- Which algorithm is faster?
- One of the assumptions of Big O is that the data set is large.
- The "grades" should be accurate tools if this is true
Running Times

Assume $N = 100,000$ and processor speed is $1,000,000,000$ operations per second

<table>
<thead>
<tr>
<th>Function</th>
<th>Running Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^N$</td>
<td>$3.2 \times 10^{30.086}$ years</td>
</tr>
<tr>
<td>$N^4$</td>
<td>3171 years</td>
</tr>
<tr>
<td>$N^3$</td>
<td>11.6 days</td>
</tr>
<tr>
<td>$N^2$</td>
<td>10 seconds</td>
</tr>
<tr>
<td>$\sqrt{N}$</td>
<td>0.032 seconds</td>
</tr>
<tr>
<td>$N \log N$</td>
<td>0.0017 seconds</td>
</tr>
<tr>
<td>$N$</td>
<td>0.0001 seconds</td>
</tr>
<tr>
<td>$\sqrt{N}$</td>
<td>$3.2 \times 10^{-7}$ seconds</td>
</tr>
<tr>
<td>$\log N$</td>
<td>$1.2 \times 10^{-8}$ seconds</td>
</tr>
</tbody>
</table>
Dykstra says: "Pictures are for the Weak."

<table>
<thead>
<tr>
<th></th>
<th>1000</th>
<th>2000</th>
<th>4000</th>
<th>8000</th>
<th>16000</th>
<th>32000</th>
<th>64000</th>
<th>128K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(N)$</td>
<td>$2.2 \times 10^{-5}$</td>
<td>$2.7 \times 10^{-5}$</td>
<td>$5.4 \times 10^{-5}$</td>
<td>$4.2 \times 10^{-5}$</td>
<td>$6.8 \times 10^{-5}$</td>
<td>$1.2 \times 10^{-4}$</td>
<td>$2.3 \times 10^{-4}$</td>
<td>$5.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$O(N \log N)$</td>
<td>$8.5 \times 10^{-5}$</td>
<td>$1.9 \times 10^{-4}$</td>
<td>$3.7 \times 10^{-4}$</td>
<td>$4.7 \times 10^{-4}$</td>
<td>$1.0 \times 10^{-3}$</td>
<td>$2.1 \times 10^{-3}$</td>
<td>$4.6 \times 10^{-3}$</td>
<td>$1.2 \times 10^{-2}$</td>
</tr>
<tr>
<td>$O(N^{3/2})$</td>
<td>$3.5 \times 10^{-5}$</td>
<td>$6.9 \times 10^{-4}$</td>
<td>$1.7 \times 10^{-3}$</td>
<td>$5.0 \times 10^{-3}$</td>
<td>$1.4 \times 10^{-2}$</td>
<td>$3.8 \times 10^{-2}$</td>
<td>$0.11$</td>
<td>$0.30$</td>
</tr>
<tr>
<td>$O(N^2)$ ind.</td>
<td>$3.4 \times 10^{-3}$</td>
<td>$1.4 \times 10^{-3}$</td>
<td>$4.4 \times 10^{-3}$</td>
<td>$0.22$</td>
<td>$0.86$</td>
<td>$3.45$</td>
<td>$13.79$</td>
<td>(55)</td>
</tr>
<tr>
<td>$O(N^2)$ dep.</td>
<td>$1.8 \times 10^{-3}$</td>
<td>$7.1 \times 10^{-3}$</td>
<td>$2.7 \times 10^{-2}$</td>
<td>$0.11$</td>
<td>$0.43$</td>
<td>$1.73$</td>
<td>$6.90$</td>
<td>(27.6)</td>
</tr>
<tr>
<td>$O(N^3)$</td>
<td>3.40</td>
<td>27.26</td>
<td>(218)</td>
<td>(1745)</td>
<td>(13,957)</td>
<td>(112k)</td>
<td>(896k)</td>
<td>(7.2m)</td>
</tr>
</tbody>
</table>

Times in Seconds. Red indicates predicated value.

CS 314 Efficiency - Complexity
# Change between Data Points

<table>
<thead>
<tr>
<th></th>
<th>1000</th>
<th>2000</th>
<th>4000</th>
<th>8000</th>
<th>16000</th>
<th>32000</th>
<th>64000</th>
<th>128K</th>
<th>256k</th>
<th>512k</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(N)</td>
<td>-</td>
<td>1.21</td>
<td>2.02</td>
<td>0.78</td>
<td>1.62</td>
<td>1.76</td>
<td>1.89</td>
<td>2.24</td>
<td>2.11</td>
<td>1.62</td>
</tr>
<tr>
<td>O(N \log N)</td>
<td>-</td>
<td>2.18</td>
<td>1.99</td>
<td>1.27</td>
<td>2.13</td>
<td>2.15</td>
<td>2.15</td>
<td>2.71</td>
<td>1.64</td>
<td>2.40</td>
</tr>
<tr>
<td>O(N^{3/2})</td>
<td>-</td>
<td>1.98</td>
<td>2.48</td>
<td>2.87</td>
<td>2.79</td>
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<td>2.85</td>
<td>2.79</td>
<td>2.82</td>
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<tr>
<td>O(N^2) ind</td>
<td>-</td>
<td>4.06</td>
<td>3.98</td>
<td>3.94</td>
<td>3.99</td>
<td>4.00</td>
<td>3.99</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>O(N^2) dep</td>
<td>-</td>
<td>4.00</td>
<td>3.82</td>
<td>3.97</td>
<td>4.00</td>
<td>4.01</td>
<td>3.98</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>O(N^3)</td>
<td>-</td>
<td>8.03</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Value obtained by $\frac{\text{Time}_x}{\text{Time}_{x-1}}$
Okay, Pictures
Put a Cap on Time

Results on a 2GhZ laptop

- $N$
- $N \log N$
- $N \sqrt{N}$
- $N^2$
No \( O(N^2) \) Data

Results on a 2Ghz laptop

![Graph showing time vs. value of N for different complexities: N, NlogN, NsqrtN.](image-url)
Just $O(N)$ and $O(N \log N)$
Just $O(N)$
## $10^9$ instructions/sec, runtimes

<table>
<thead>
<tr>
<th>$N$</th>
<th>$O(\log N)$</th>
<th>$O(N)$</th>
<th>$O(N \log N)$</th>
<th>$O(N^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.00000000003</td>
<td>0.000000001</td>
<td>0.0000000033</td>
<td>0.0000001</td>
</tr>
<tr>
<td>100</td>
<td>0.00000000007</td>
<td>0.000000010</td>
<td>0.0000000664</td>
<td>0.0001000</td>
</tr>
<tr>
<td>1,000</td>
<td>0.0000000010</td>
<td>0.000001000</td>
<td>0.000100000</td>
<td>0.001</td>
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<tr>
<td>10,000</td>
<td>0.0000000013</td>
<td>0.000010000</td>
<td>0.001329000</td>
<td>0.1 min</td>
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<td>100,000</td>
<td>0.0000000017</td>
<td>0.000100000</td>
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<td>10 sec</td>
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<tr>
<td>1,000,000</td>
<td>0.0000000020</td>
<td>0.001</td>
<td>0.0199</td>
<td>16.7 min</td>
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<tr>
<td>1,000,000,000</td>
<td>0.0000000030</td>
<td>1.0 sec</td>
<td>30 sec</td>
<td>31.7 years</td>
</tr>
</tbody>
</table>
Formal Definition of Big O (repeated)

- $T(N)$ is $O( F(N) )$ if there are positive constants $c$ and $N_0$ such that $T(N) \leq cF(N)$ when $N \geq N_0$
  - $N$ is the size of the data set the algorithm works on
  - $T(N)$ is a function that characterizes the actual running time of the algorithm
  - $F(N)$ is a function that characterizes an upper bounds on $T(N)$. It is a limit on the running time of the algorithm
  - $c$ and $N_0$ are constants
More on the Formal Definition

- There is a point $N_0$ such that for all values of $N$ that are past this point, $T(N)$ is bounded by some multiple of $F(N)$
- Thus if $T(N)$ of the algorithm is $O(N^2)$ then, ignoring constants, at some point we can *bound* the running time by a quadratic function.
- Given a *linear* algorithm it is *technically correct* to say the running time is $O(N^2)$. $O(N)$ is a more precise answer as to the Big O of the linear algorithm
  - Thus the caveat “pick the most restrictive function” in Big O type questions.
What it All Means

- $T(N)$ is the actual growth rate of the algorithm
  - can be equated to the number of executable statements in a program or chunk of code
- $F(N)$ is the function that bounds the growth rate
  - may be upper or lower bound
- $T(N)$ may not necessarily equal $F(N)$
  - constants and lesser terms ignored because it is a bounding function
Other Algorithmic Analysis Tools

- **Big Omega** $T(N)$ is $\Omega(F(N))$ if there are positive constants $c$ and $N_0$ such that $T(N) \geq cF(N)$ when $N \geq N_0$
  
  - Big O is similar to less than or equal, an upper bound
  
  - Big Omega is similar to greater than or equal, a lower bound

- **Big Theta** $T(N)$ is $\Theta(F(N))$ if and only if $T(N)$ is $O(F(N))$ and $T(N)$ is $\Omega(F(N))$.
  
  - Big Theta is similar to equals
### Relative Rates of Growth

<table>
<thead>
<tr>
<th>Analysis Type</th>
<th>Mathematical Expression</th>
<th>Relative Rates of Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big O</td>
<td>$T(N) = O(F(N))$</td>
<td>$T(N) \leq F(N)$</td>
</tr>
<tr>
<td>Big $\Omega$</td>
<td>$T(N) = \Omega(F(N))$</td>
<td>$T(N) \geq F(N)$</td>
</tr>
<tr>
<td>Big $\theta$</td>
<td>$T(N) = \theta(F(N))$</td>
<td>$T(N) = F(N)$</td>
</tr>
</tbody>
</table>

"In spite of the additional precision offered by Big Theta, Big O is more commonly used, except by researchers in the algorithms analysis field" - Mark Weiss