CS 343: Artificial Intelligence

Markov Decision Processes II

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[These slides based on those of Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]
Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent’s path

- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put

- The agent receives rewards each time step
  - Small “living” reward each step (can be negative)
  - Big rewards come at the end (good or bad)

- Goal: maximize sum of (discounted) rewards
Recap: MDPs

- Markov decision processes:
  - States $S$
  - Actions $A$
  - Transitions $P(s'|s,a)$ (or $T(s,a,s')$)
  - Rewards $R(s,a,s')$ (and discount $\gamma$)
  - Start state $s_0$

- Quantities:
  - Policy = map of states to actions
  - Utility = sum of discounted rewards
  - Values = expected future utility from a state (max node)
  - Q-Values = expected future utility from a q-state (chance node)
Optimal Quantities

- The value (utility) of a state $s$:
  \[ V^*(s) = \text{expected utility starting in } s \text{ and acting optimally} \]

- The value (utility) of a q-state $(s,a)$:
  \[ Q^*(s,a) = \text{expected utility starting out having taken action } a \text{ from state } s \text{ and (thereafter) acting optimally} \]

- The optimal policy:
  \[ \pi^*(s) = \text{optimal action from state } s \]
Gridworld Values $V^*$

VALUES AFTER 100 ITERATIONS
Gridworld: $Q^*$

Q-VALUES AFTER 100 ITERATIONS
The Bellman Equations

How to be optimal:

Step 1: Take correct first action
Step 2: Keep being optimal
The Bellman Equations

- Definition of “optimal utility” via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values:

\[ V^*(s) = \max_a Q^*(s, a) \]

\[ Q^*(s, a) = \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]

\[ V^*(s) = \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]

- These are the Bellman equations, and they characterize optimal values in a way we’ll use over and over.
Value Iteration

- Bellman equations **characterize** the optimal values:
  \[ V^*(s) = \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]

- Value iteration **computes** them:
  \[ V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_k(s') \right] \]

- Value iteration is just a fixed point solution method
  - though the \( V_k \) vectors are also interpretable as time-limited values
Convergence*

- How do we know the $V_k$ vectors are going to converge?

  - Case 1: If the tree has maximum depth $M$, then $V_M$ holds the actual untruncated values

  - Case 2: If the discount is less than 1
    - Sketch: For any state $V_k$ and $V_{k+1}$ can be viewed as depth $k$ and $k+1$ expectimax, which result in nearly identical search trees
    - The difference is that on the bottom layer, $V_{k+1}$ has actual rewards while $V_k$ has zeros
    - That last layer is at best all $R_{\text{MAX}}$
    - It is at worst $R_{\text{MIN}}$
    - But everything is discounted by $\gamma^k$ that far out
    - So $V_k$ and $V_{k+1}$ are at most $\gamma^k \max |R|$ different
    - So as $k$ increases, the values converge
Policy Methods
Policy Evaluation
Fixed Policies

- Expectimax trees max over all actions to compute the optimal values.
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state,
  - though the tree’s value would depend on which policy we fixed.

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**Diagram:**
- Do the optimal action
- Do what $\pi$ says to do
Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state $s$ under a fixed (generally non-optimal) policy

- Define the utility of a state $s$, under a fixed policy $\pi$:
  \[ V^\pi(s) = \text{expected total discounted rewards starting in } s \text{ and following } \pi \]

- Recursive relation (one-step look-ahead / Bellman equation):
  \[ V^\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^\pi(s')] \]
Example: Policy Evaluation

Always Go Right

Always Go Forward
Example: Policy Evaluation

Always Go Right

Always Go Forward
Policy Evaluation

- How do we calculate the V’s for a fixed policy $\pi$?

- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

  \[
  V_0^\pi(s) = 0 \\
  V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_k^\pi(s')] 
  \]

  - Efficiency: $O(S^2)$ per iteration

- Idea 2: Without the maxes, the Bellman equations are just a linear system
  - Solve with Matlab (or your favorite linear system solver)
Policy Extraction
Computing Actions from Values

- Let’s imagine we have the optimal values $V^*(s)$

- How should we act?
  - It’s not obvious!

- We need to do a mini-expectimax (one step)

$$
\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')] 
$$

- This is called policy extraction, since it gets the policy implied by the values
Computing Actions from Q-Values

- Let’s imagine we have the optimal q-values:

- How should we act?
  - Completely trivial to decide!

\[ \pi^*(s) = \arg \max_{\alpha} Q^*(s, \alpha) \]

- Important lesson: actions are easier to select from q-values than values!
- In fact, you don’t even need a model!
Policy Iteration
Problems with Value Iteration

- Value iteration repeats the Bellman updates:
  \[ V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_k(s') \right] \]

- Problem 1: It’s slow – \( O(S^2A) \) per iteration

- Problem 2: The “max” at each state rarely changes

- Problem 3: The policy often converges long before the values
$k=0$

Noise = 0.2
Discount = 0.9
Living reward = 0

VALUES AFTER 0 ITERATIONS
\( k = 1 \)

VALUES AFTER 1 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
$k = 2$

VALUES AFTER 2 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
$k=3$

Noise = 0.2
Discount = 0.9
Living reward = 0

VALUES AFTER 3 ITERATIONS
k=4

VALUES AFTER 4 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
<p>| | | | |</p>
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<td>0.22</td>
<td>0.37</td>
<td>0.13</td>
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</table>

VALUES AFTER 5 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=6

VALUES AFTER 6 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=7

Values after 7 iterations:

- Top left: 0.62
- Top middle: 0.74
- Top right: 0.85
- Top right corner: 1.00
- Middle left: 0.50
- Middle middle: 0.57
- Middle right: -1.00
- Bottom left: 0.34
- Bottom middle: 0.36
- Bottom right: 0.45
- Bottom right corner: 0.24

Noise = 0.2
Discount = 0.9
Living reward = 0
k=8

VALUES AFTER 8 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=9

VALUES AFTER 9 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
$k=10$

VALUES AFTER 10 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=11

VALUES AFTER 11 ITERATIONS

Noise = 0.2  Discount = 0.9  Living reward = 0
k=12

VALUES AFTER 12 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=100

VALUES AFTER 100 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
Policy Iteration

- Alternative approach for optimal values:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

- This is policy iteration
  - It’s still optimal!
  - Can converge (much) faster under some conditions
Policy Iteration

- Evaluation: For fixed current policy $\pi$, find values with policy evaluation:
  - Iterate until values converge:
    \[
    V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[ R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s') \right]
    \]

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:
    \[
    \pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^{\pi_i}(s') \right]
    \]
Both value iteration and policy iteration compute the same thing (all optimal values)

In value iteration:
- Every iteration updates both the values and (implicitly) the policy
- We don’t track the policy, but taking the max over actions implicitly recomputes it

In policy iteration:
- We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
- After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
- The new policy will be better (or we’re done)

Both are dynamic programs for solving MDPs
Summary: MDP Algorithms

- **So you want to….**
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)

- **These all look the same!**
  - They basically are – they are all variations of Bellman updates
  - They all use one-step lookahead expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions
Double Bandits
Double-Bandit MDP

- Actions: Blue, Red
- States: Win, Lose

No discount
100 time steps
Both states have the same value.
Solving MDPs is offline planning

- You determine all quantities through computation
- You need to know the details of the MDP
- You do not actually play the game!

<table>
<thead>
<tr>
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<th>Value</th>
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<tbody>
<tr>
<td>Play Red</td>
<td>150</td>
</tr>
<tr>
<td>Play Blue</td>
<td>100</td>
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No discount
100 time steps
Both states have the same value
Let’s Play!

$1 \text{ WINNER}$

$777$

$2 \ 2 \ 0 \ 2 \ 2$

$2 \ 2 \ 0 \ 0 \ 0 \ 0$

Double $2/0$ or Nothing
Online Planning

- Rules changed! Red’s win chance is different.
Let’s Play!
What Just Happened?

- That wasn’t planning, it was learning!
  - Specifically, reinforcement learning
  - There was an MDP, but you couldn’t solve it with just computation
  - You needed to actually act to figure it out

- Important ideas in reinforcement learning that came up
  - Exploration: you have to try unknown actions to get information
  - Exploitation: eventually, you have to use what you know
  - Regret: even if you learn intelligently, you make mistakes
  - Sampling: because of chance, you have to try things repeatedly
  - Difficulty: learning can be much harder than solving a known MDP
Next Time: Reinforcement Learning!