Particle Filters and Applications of HMMs
Recap: Reasoning Over Time

- Markov models

\[ P(X_1) \quad P(X|X_{-1}) \]

- Hidden Markov models

\[
\begin{array}{c}
X_1 \quad X_2 \quad X_3 \quad X_4 \\
E_1 \quad E_2 \quad E_3 \quad E_4
\end{array}
\]

\[
P(E|X)
\]

<table>
<thead>
<tr>
<th>X</th>
<th>E</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>rain</td>
<td>umbrella</td>
<td>0.9</td>
</tr>
<tr>
<td>rain</td>
<td>no umbrella</td>
<td>0.1</td>
</tr>
<tr>
<td>sun</td>
<td>umbrella</td>
<td>0.2</td>
</tr>
<tr>
<td>sun</td>
<td>no umbrella</td>
<td>0.8</td>
</tr>
</tbody>
</table>
Recap: Forward Algo - Passage of Time

- Assume we have current belief \( P(X \mid \text{evidence to date}) \)
  \[
  B(X_t) = P(X_t \mid e_{1:t})
  \]

- Then, after one time step passes:
  \[
  P(X_{t+1} \mid e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t \mid e_{1:t})
  \]
  \[
  = \sum_{x_t} P(X_{t+1} \mid x_t, e_{1:t}) P(x_t \mid e_{1:t})
  \]
  \[
  = \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})
  \]

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step \( t \) the belief is about, and what evidence it includes

- Or compactly:
  \[
  B'(X_{t+1}) = \sum_{x_t} P(X' \mid x_t) B(x_t)
  \]
Recap: Forward Algo - Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)
Recap: Forward Algo - Observation

- Assume we have current belief $P(X \mid \text{previous evidence})$:
  $$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

- Then, after evidence comes in:
  $$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})}$$
  $$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} \mid e_{1:t})$$
  $$= P(e_{t+1} \mid e_{1:t}, X_{t+1}) P(X_{t+1} \mid e_{1:t})$$
  $$= P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

- Or, compactly:
  $$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} \mid X_{t+1}) B'(X_{t+1})$$

- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize
Recap: Forward Algo - Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

\[ B(X) \propto P(e \mid X) B'(X) \]
Recap: The Forward Algorithm

- We are given evidence at each time and want to know

\[ B_t(X) = P(X_t | e_{1:t}) \]

- We can derive the following updates

\[
P(x_t | e_{1:t}) \propto_X P(x_t, e_{1:t}) \\
= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) \\
= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t) \\
= P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1}, e_{1:t-1})
\]

We can normalize as we go if we want to have \( P(x|e) \) at each time step, or just once at the end...
Recap: Online Filtering w/ Forward Algo

Elapse time: compute $P(X_t \mid e_{1:t-1})$

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$

Observe: compute $P(X_t \mid e_{1:t})$

$$P(x_t | e_{1:t}) \propto P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$

Belief: $<$P(rain), P(sun)$>$

- $P(X_1)$
  - Prior on $X_1$
  - $P(X_1 \mid E_1 = \text{umbrella})$
  - Observe

- $P(X_2 \mid E_1 = \text{umbrella})$
  - Elapse time

- $P(X_2 \mid E_1 = \text{umb}, E_2 = \text{umb})$
  - Observe
Particle Filtering
Particle Filtering

- Filtering: approximate solution

- Sometimes $|X|$ is too big to use exact inference
  - $|X|$ may be too big to even store $B(X)$
  - E.g. $X$ is continuous

- Solution: approximate inference
  - Track samples of $X$, not all values
  - Samples are called particles
  - Time per step is linear in the number of samples
  - But: number needed may be large
  - In memory: list of particles, not states

- This is how robot localization works in practice

- Particle is just new name for sample
Our representation of $P(X)$ is now a list of $N$ particles (samples)
- Generally, $N << |X|$ (...but not in project 4)
- Storing map from $X$ to counts would defeat the point

$P(x)$ approximated by number of particles with value $x$
- So, many $x$ may have $P(x) = 0!$
- More particles, more accuracy

For now, all particles have a weight of 1

Particle filtering uses three repeated steps:
- Elapse time and observe (similar to exact filtering) and resample
Example: Elapse Time

Belief over possible ghost positions at time $t$

Elapse Time

Policy: ghosts always move up (or stay in place if already at top)

New belief at time $t+1$
Example: Elapse Time

Belief over possible ghost positions at time $t$

Elapse Time

Policy: ghosts always move up (or stay in place if already at top)

New belief at time $t+1$
Particle Filtering: Elapse Time

- Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

- Sample frequencies reflect the transition probabilities

- Here, most samples move clockwise, but some move in another direction or stay in place

- This captures the passage of time
  - If enough samples, close to exact values before and after (consistent)
Example: Observe

<table>
<thead>
<tr>
<th></th>
<th>0.5</th>
<th>0.4</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

Belief over possible ghost positions before observation

Observation and evidence likelihoods $p(e | X)$

New belief after observation
Example: Observe

Belief over possible ghost positions before observation

Observation and evidence likelihoods $p(e | X)$

New belief after observation

<p>| | | |</p>
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</tr>
<tr>
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<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Observation: 0.4

New belief: 0.1
- Slightly trickier:
  - Don’t sample observation, fix it
  - Similar to likelihood weighting, downweight samples based on the evidence
    \[ w(x) = P(e|x) \]
    \[ B(X) \propto P(e|X)B'(X) \]
  - As before, the probabilities don’t sum to one, since all have been downweighted
Particle Filtering: Resample

- Rather than tracking weighted samples, we resample.

- N times, we choose from our weighted sample distribution (i.e. draw with replacement).

- This essentially renormalizes the distribution.

- Now the update is complete for this time step, continue with the next one.
Recap: Particle Filtering

- **Particles**: track samples of states rather than an explicit distribution

<table>
<thead>
<tr>
<th>Elapse</th>
<th>Weight</th>
<th>Resample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particles:</td>
<td>Particles:</td>
<td>Particles:</td>
</tr>
<tr>
<td>(3,3)</td>
<td>(3,2) w=.9</td>
<td>(3,2)</td>
</tr>
<tr>
<td>(2,3)</td>
<td>(2,3) w=.2</td>
<td>(2,3)</td>
</tr>
<tr>
<td>(3,3)</td>
<td>(3,2) w=.9</td>
<td>(3,2)</td>
</tr>
<tr>
<td>(3,2)</td>
<td>(3,1) w=.4</td>
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<td>(3,2) w=.9</td>
<td>(3,2)</td>
</tr>
<tr>
<td>(1,2)</td>
<td>(1,3) w=.1</td>
<td>(1,3)</td>
</tr>
<tr>
<td>(3,3)</td>
<td>(2,3) w=.2</td>
<td>(2,3)</td>
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</tr>
<tr>
<td>(2,3)</td>
<td>(2,2) w=.4</td>
<td>(2,2)</td>
</tr>
</tbody>
</table>

| (New) Particles: |
| (3,2) |
| (2,2) |
| (3,2) |
| (2,3) |
| (3,3) |
| (3,2) |
| (2,3) |
| (3,3) |
| (2,3) |

- **Weights**: assign weights to each particle based on the likelihood of the observations.
- **Resampling**: select particles with replacement according to their weights.

Diagram showing the process of particle filtering with a series of grids representing the elapse, weight, and resample steps.
Moderate Number of Particles
One Particle
Huge Number of Particles
Robot Localization

- In robot localization:
  - We know the map, but not the robot’s position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store $B(X)$
  - Particle filtering is a main technique
Particle Filter Localization (Sonar)

Global localization with sonar sensors
Robot Mapping

- **SLAM: Simultaneous Localization And Mapping**
  - We do not know the map or our location
  - State consists of position AND map!
  - Main techniques: Kalman filtering (Gaussian HMMs) and particle methods

DP-SLAM, Ron Parr
Particle Filter SLAM – Video 1
Dynamic Bayes Nets
Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net structure at each time
- Variables from time $t$ can condition on those from $t-1$

- Dynamic Bayes nets are a generalization of HMMs
Exact Inference in DBNs

- Variable elimination applies to dynamic Bayes nets
- Procedure: “unroll” the network for $T$ time steps, then eliminate variables until $P(X_t|e_{1:T})$ is computed
- Online belief updates: Eliminate all variables from the previous time step; store factors for current time only
A particle is a complete sample for a time step

**Initialize**: Generate prior samples for the $t=1$ Bayes net
- Example particle: $G_1^a = (3,3) \quad G_1^b = (5,3)$

**Elapse time**: Sample a successor for each particle
- Example successor: $G_2^a = (2,3) \quad G_2^b = (6,3)$

**Observe**: Weight each *entire* sample by the likelihood of the evidence conditioned on the sample
- Likelihood: $P(E_1^a | G_1^a) \times P(E_1^b | G_1^b)$

**Resample**: Select samples (tuples of values) in proportion to their likelihood (weight)
Most Likely Explanation
HMMs: MLE Queries

- HMMs defined by
  - States $X$
  - Observations $E$
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X|X_{-1})$
  - Emissions: $P(E|X)$

- New query: most likely explanation: $\arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$

- New method: the Viterbi algorithm

- Question: Why not just apply filtering and predict most likely value of each variable separately?
State Trellis

- State trellis: graph of states and transitions over time

- Each arc represents some transition $x_{t-1} \rightarrow x_t$
- Each arc has weight $P(x_t|x_{t-1})P(e_t|x_t)$
- Each path is a sequence of states
- The product of weights on a path is that sequence’s probability along with the evidence
- Forward algorithm computes sums of all paths to each node, Viterbi computes best paths
- Exponentially many paths, but dynamic programming can find best path in linear time!
Forward / Viterbi Algorithms

Forward Algorithm (Sum)

\[ f_t[x_t] = P(x_t, e_{1:t}) \]

\[ = P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) f_{t-1}[x_{t-1}] \]

Viterbi Algorithm (Max)

\[ m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t}) \]

\[ = P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}] \]
Speech Recognition
Speech Recognition in Action
Digitizing Speech
Speech waveforms

- Speech input is an acoustic waveform

Figure: Simon Arnfield, http://www.psyc.leeds.ac.uk/research/cogn/speech/tutorial/
Spectral Analysis

- Frequency gives pitch; amplitude gives volume
  - Sampling at ~8 kHz (phone), ~16 kHz (mic) (kHz=1000 cycles/sec)

- Fourier transform of wave displayed as a spectrogram
  - Darkness indicates energy at each frequency
- Time slices are translated into acoustic feature vectors (~39 real numbers per slice)

- These are the observations $E$, now we need the hidden states $X$
Speech State Space

- **HMM Specification**
  - $P(E|X)$ encodes which acoustic vectors are appropriate for each phoneme (each kind of sound)
  - $P(X|X')$ encodes how sounds can be strung together

- **State Space**
  - We will have one state for each sound in each word
  - Mostly, states advance sound by sound
  - Build a little state graph for each word and chain them together to form the state space $X$
States in a Word
Transitions with a Bigram Model

Training Counts

<table>
<thead>
<tr>
<th>Count</th>
<th>Word Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td>198015222</td>
<td>the first</td>
</tr>
<tr>
<td>194623024</td>
<td>the same</td>
</tr>
<tr>
<td>168504105</td>
<td>the following</td>
</tr>
<tr>
<td>158562063</td>
<td>the world</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>14112454</td>
<td>the door</td>
</tr>
<tr>
<td>23135851162</td>
<td>*</td>
</tr>
</tbody>
</table>

\[
P(\text{door}|\text{the}) = \frac{14112454}{23135851162} = 0.0006
\]
Decoding (Viterbi)

- Finding the words given the acoustics is an HMM inference problem.
- Which state sequence $x_{1:T}$ is most likely given the evidence $e_{1:T}$?

$$x_{1:T}^* = \arg \max_{x_{1:T}} P(x_{1:T}|e_{1:T}) = \arg \max_{x_{1:T}} P(x_{1:T}, e_{1:T})$$

- From the sequence $x$, we can simply read off the words.