Defining functions recursively

Most of the time, you'll be given a function that is defined recursively, and you'll be asked to find a closed form solution for the function. For example, a function $g(n)$ defined on the positive integers by $g(1) = 3$ and $g(n) = n \cdot g(n-1)$ has a closed form solution of $g(n) = 3 \cdot n!$.

Some of the questions in Homework #2 ask you to find closed form solutions to recursively defined functions, and prove your solutions correct by induction. However, there are also questions in Homework #2, in which I ask you to define a function recursively, while the function is given a closed form.

Here's what I mean by this.

Suppose the function $f(n)$ is defined for all positive integers n by $f(n) = 5$. Then this function can be defined recursively as follows:

- $f(1) = 5$, and $f(n) = f(n-1)$.

Another example is the function $g(n) = 3 + 6n$, defined on the positive integers. Then you could define $g(n)$ recursively by

- $g(1) = 9$ and $g(n) = g(n-1) + 6$.

And, finally, here's another. Suppose the function $h(n)$ is defined on the positive integers by $h(n) = 3^n$. Then $h(n)$ can be defined recursively by

- $h(1) = 3$ and $h(n) = 3 \cdot h(n-1)$.

The way to *derive* these recursive functions is to see if you can express the function in terms of smaller values. For example, if you are asked to define a function $f(n)$ recursively, you might try subtracting $f(n-1)$ from $f(n)$, or dividing $f(n)$ by $f(n-1)$.

Trying that approach for these functions above, we find the following:

- When $f(n) = 5$, then $f(n-1) = 5$, and so $f(n) - f(n-1) = 0$. And so $f(n) = f(n-1)$. The initial value is $f(1) = 5$.

- When $g(n) = 3 + 6n$, then $g(n-1) = 3 + 6(n-1) = 3 + 6n - 6 = 6n - 3$. Therefore $g(n) - g(n-1) = 6$, and so $g(n) = g(n-1) + 6$. The initial value is $g(1) = 9$.

- When $h(n) = 3^n$, then $h(n-1) = 3^{n-1}$. Here subtracting yields $h(n) - h(n-1) = 3^n - 3^{n-1}$, and so $h(n) = h(n-1) + 3^n - 3^{n-1}$. Kind of not very satisfying. But, dividing works! We get $h(n)/h(n-1) = 3$, and so $h(n) = 3 \cdot h(n-1)$. The initial value is $h(1) = 3$.

Try the homeworks now.