Homework #2, Solutions, Spring 2009

Problems:

1. Phrase each of the following real world problems as a graph-theoretic problem. Make sure to specify what the graph consists of (what are the vertices and what are the edges), and what the output consists of (described explicitly in terms of the graph at that point). Use English to explain what you mean, as carefully and unambiguously as you can. Refer to the notes for the correct use of graph-theoretic language! Use mathematics as well, if you can, to add precision (but do that separately).

   (a) *You are given a collection of people, and you know which people know each other (which you assume is symmetric). You want to find a group of people who, between them, know everyone. You'd like this group to be as small as possible.*

   Answer: The input to the problem is a graph $G = (V, E)$ in which the vertex set V consists of all the people in the collection, and E consists of the pairs (a, b) of people which know each other. Your objective is a smallest subset V' of V so that for all vertices $w \in V - V'$, there is some vertex in V' adjacent to v. In mathematics: your objective is a subset $V' \subseteq V$ such that $\forall v \in V - V', \exists y \in V'$ *s.t.* $(v, y) \in E$, and such that $|V'|$ is as small as possible subject to that constraint.

   (b) *It is 1900, and you are the match maker for a small village in the Ukraine. Thus, your job is to marry off all the unmarried young men and women. You know which pairs of men and women are willing to be married (and, more importantly, whether their parents are willing to let their children get married, which depends also upon the dowry and other factors). You get a fixed fee for each marriage you make. Under the assumptions that you cannot marry anyone to more than one person and you cannot perform any same sex marriages, you'd like to maximize the fee you'll be paid.*

   Answer: Your input is a graph $G = (V, E)$ in which the vertices in V represent the unmarried young men and women, and there is an edge between every young man and woman who are willing to marry, and whose parents don't object to the marriage. Your objective is a subset E' of E of maximum cardinality so that no two edges in E' share an endpoint. Mathematically, your objective is a subset $E' \subseteq E$ such that $\forall \{e, e'\} \subseteq E', e \cap e' = \emptyset$, and such that $|E'|$ is maximum subject to this constraint. (Note that in this definition, we inherently are using the denotation of edges as sets of two vertices, so that we can replace a cumbersome way of saying that two edges share no endpoints in a simple way.)

   (c) *You are the social organizer for your friends' graduations, and you need to arrange a number of graduation parties. The only problem is* *that some people can't stand each other, and so you can't have them at the same party. Also, you can only afford to throw k parties (and k is part of the input). Determine if you can afford to arrange parties for everyone.*

Answer: the graph $G = (V, E)$ has a vertex for each of your friends, and edges (x, y) where x and y can't stand each other. Your objective is a partition of V (if it exists) into at most k pairwise-disjoint subsets $V_1, V_2, \ldots, V_i$ (with $i \leq k$) so that for each set V_a no two vertices are adjacent. Mathematically, you seek sets $V_1, V_2, \ldots, V_i$, with $i \leq k$, such that when $a \neq b$, $V_a \cap V_b = \emptyset, \cup V_a = V$, and such that for all a and for all $\{x, y\} \subseteq V_a$, $(x, y) \notin E$.

2. Show how you'd use an oracle for the Yes/No problem to construct a solution to each of the construction problems below.

   (a) *Given a graph, determine if it has a clique of size 5, and return it if it does. (The oracle takes as input the graph $G = (V, E)$ and the size k of the desired clique, and tells you if the graph has a clique of that size.)*

   Answer: begin by asking the oracle if the graph $G = (V, E)$ has a clique of size 5. If no, return "No such clique", and exit. If yes, then order the vertices $v_1, v_2, \ldots, v_n$.

   For $i = 1, 2, \ldots, n$, do the following:

   - Ask the oracle if $G - \{v_i\}$ has a clique of size 5. If so, remove v_i from G.

   The graph G at the end of this will consist of exactly 5 vertices, and will constitute a clique. Return the vertex set of G.

   (b) *Given a set X of items of different weights, and given a specific target total weight B, determine if there is a subset of X of total weight B, and* return the subset *if it exists. For example, if the input has set $X = \{1, 5, 8, 13, 21, 27, 30\}$ and $B = 41$, then the answer is* yes, *and the subset would be $\{1, 13, 27\}$. (The oracle takes the set X and the target weight B, and tells you if the set X has a subset of that total weight.)*

   Answer: First ask the oracle to answer the question for the given input (X, B). If the oracle says "No", return "no" and exit; otherwise, continue. Order the elements of X as $x_1, x_2, \ldots, x_n$, where $n = |X|$. Let $X[i, i+1, \ldots, n]$ denote the set $\{x_i, x_{i+1}, \ldots, x_n\}$.

   For $i = 1, 2, \ldots, n$ DO:

   - If Oracle(X[i+1 . . . n],B) then remove x_i from X.

   (At the end X will be a subset for which the sum of its elements is the target weight B.) Return X.

(c) *Given a sequence of integers* $a_1, a_2, \ldots, a_n$*, find an increasing subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ *(so* $i_1 < i_2 < \ldots i_k$*) of maximum length* k*. Thus, the solution to* $1, 2, 5, 3, 4, 8, 5, 1, 9$ *is* $1, 2, 3, 4, 5, 9$*, which has length 6. (The oracle takes the sequence of integers as input, and tells you the maximum length of an increasing subsequence.)*

Answer: First find the maximum length L of an increasing subsequence. For a sequence $B = b_1, b_2, \ldots, b_m$, let $B - b_i$ denote the sequence $b_1, b_2, \ldots, b_i, b_{i+1}, \ldots, b_n$ (i.e., you delete the entry b_i from B). For $i = 1, 2, \ldots, n$, do the following:

- If $Oracle(A - a_i) = L$ then delete a_i from A (i.e., replace A by $A - a_i$).

(d) (Extra Credit) *Given a graph, determine if it has a 3-coloring of the vertices (that is a coloring of the vertices so that no two adjacent vertices have the same color), and return one if it does. (The oracle would take the graph as input, and tell you if it has a 3-coloring.)*

Answer: will provide in class.

3. Express each of the following functions recursively (i.e., give the value for $p(1)$ and define $p(n)$ in terms of $p(n-1)$):

   (a) $p(n) = n^2$. Answer: $p(1) = 1$ and $p(n) = p(n-1) + 2n - 1$ when $n > 1$.

   (b) $p(n) = 3^n$. Answer: $p(1) = 3$ and $p(n) = 3p(n-1)$ when $n > 1$.

   (c) $p(n) = (n+1)!$ Answer: $p(1) = 1$ and $p(n) = (n+1)p(n-1)$ when $n > 1$.

   (d) $p(n) = n + 5$ Answer: $p(1) = 6$ and $p(n) = 1 + p(n-1)$.

4. *Pick any* one *of the functions from the previous problem, and prove that it equals its recursive definition, using induction.*

   Answer: for $p(n) = n + 5$, we will prove $p(n)$ satisfies the recurrence given above. Let $q(n)$ be defined by $q(1) = 6$ and $q(n) = q(n-1) + 1$. We will prove that $p(n) = q(n)$ for all integers $n \geq 1$ by induction on n. The base case is $n = 1$, for which $p(n) = 1 = q(n)$. The indutive hypothesis is that $p(X) = q(X)$ for some particular $X \geq 1$. Then

   $q(X+1) = q(X) + 1$ by definition of $q(X)$
   $=p(X) + 1$ by the Inductive Hypothesis
   $=X + 5 + 1$ by the definition of $p(X)$
   $=X + 6$
   $=(X+1) + 5$
   $=p(X+1)$ by definition of $p(X+1)$
   q.e.d.

5. For each of the following pairs of functions $f(n)$ and $g(n)$, state (without proof) which of the following cases is true: (a) $f(n)$ is $O(g(n))$, (b) $g(n)$

is $O(f(n))$, (c) each is big-oh of the other (i.e., $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$), or (d) neither is big-oh of the other.

(a) $f(n) = 3n^2, g(n) = 5n^3$. (a)

(b) $f(n) = \log n, g(n) = 500$. (b)

(c) $f(n) = (\log n)^5, g(n) = n/2$. (a)

(d) $f(n) = \sqrt{n}, g(n) = \log n$. (b)

(e) $f(n) = \sqrt{n}, g(n) = n$. (a)

(f) $f(n) =$

- 1 if n is odd
- n if n is even

$g(n) = n$. (a)

6. *Prove that* $3n^2$ *is* $O(n^2 + 100)$ *(by producing the two constants).*

Answer: We will show that $3n^2$ is $O(n^2 + 100)$ by proving that $3n^2 \leq C_1(n^2 + 100)$ for all $n \geq C_2$, where $C_1 = 3$ and $C_2 = 1$.

$3n^2 \leq 1 \cdot 3(n^2)$ for all n since $n^2 \geq 0$
$\leq 3(n^2) + 300$
$= 3(n^2 + 100)$

Thus $3n^2 \leq 3(n^2 + 100)$ for all n. Therefore, in particular the statement holds for all $n \geq 1$.

q.e.d.