CS345H: Programming Languages

Lecture 8: Operational Semantics II

Thomas Dillig
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Will look at one more formalism for specifying meaning today
We are still missing semantics for key constructs in the L programming language
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Let’s start with the if expression: if $e_1$ then $e_2$ else $e_3$. 
Back to Operational Semantics

- We are still missing semantics for key constructs in the L programming language.

- Let’s start with the *if expression*: if e1 then e2 else e3.

- Recall meaning: If e1 evaluates to a non-zero integer, the meaning of the expression is e2, otherwise e3.
We are still missing semantics for key constructs in the L programming language

Let’s start with the if expression: if e1 then e2 else e3.

Recall meaning: If e1 evaluates to a non-zero integer, the meaning of the expression is e2, otherwise e3

Any ideas on how to write this as an operational semantics rule?
Operational Semantics of Conditionals

- **Difficulty:** What happens depends on whether $e_1$ evaluates to 0 or not.
Operational Semantics of Conditionals

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- **Solution:** Write two rules, one for the case where e1 evaluates to 0 and one for the case where e1 evaluates to a non-zero integer.
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- What if $e_1$ evaluates to 0?
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- **Solution:** Write two rules, one for the case where \( e_1 \) evaluates to 0 and one for the case where \( e_1 \) evaluates to a non-zero integer.

- **What if \( e_1 \) evaluates to 0?**

  \[
  \begin{align*}
  E & \vdash e_1 : 0 \\
  E & \vdash e_3 : e' \\
  E & \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : e'
  \end{align*}
  \]
Operational Semantics of Conditionals Cont.

- What if $e_1$ evaluates to a non-zero integer?
Operational Semantics of Conditionals Cont.

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$$
\begin{align*}
E & \vdash e_1 : \text{non-zero integer} \\
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\end{align*}
$$

Upshot: Can encode choice by giving multiple rules for same construct.

But need to make sure at most one rule can apply at any point for deterministic semantics.

Deterministic Semantics: Every program evaluates to at most one value.
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▶ **Deterministic Semantics**: Every program evaluates to at most one value
Operational Semantics of Function Definitions

► **Recall:** In L, function definitions of the form
  \[ \text{fun } f \text{ with } x_1, \ldots, x_n = e \text{ in } \ldots \] are equivalent to
  \[ \text{let } f = \lambda x_1 \ldots \lambda x_n. e \text{ in } \ldots \]
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  `fun f with x1,...,xn=e in...`

  are equivalent to

  `let f = lambda x1...lambda xn.e in ...`

- To define the meaning of a function definition, we can either repeat the lambda and let binding rules in one rule or rewrite the function definition into let and lambda’s and invoke the existing rules.

This only works if there are no circular reductions!
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- To define the meaning of a function definition, we can either repeat the lambda and let binding rules in one rule or rewrite the function definition into let and lambda’s and invoke the existing rules.

- We will do the latter:
  \[
  \frac{E \vdash \text{let } f = \text{lambda } x_1 \ldots \text{lambda } x_n . e_1 \text{ in } e_2 : e}{E \vdash \text{fun } f \text{ with } x_1, \ldots, x_n = e_1 \text{ in } e_2 : e}
  \]
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The trick we just used to give meaning to function definitions is also useful for giving meaning to variable-length expressions.
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Consider the following grammar for a list of integers:

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S \rightarrow [E] \\
E \rightarrow \text{int } E | \text{int}
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Operational Semantics of Variable-Length Expressions

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- Consider the following grammar for a list of integers:

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- Example strings in L(S): [3], [2 3 4], [1 3], …
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E \rightarrow \text{int } E \mid \text{int}
\]

- Example strings in $L(S)$: $[3], [2 \ 3 \ 4], [1 \ 3], \ldots$

- Suppose we want to define the meaning of a list of integers as their sum: How can we write operational semantics for this mini-language?
Observation: Difficulty caused by unknown length of list

Solution: Think recursively! The sum of a list of k integers can be obtained by removing the first integer, computing the sum of the remainder and adding the two values.

This translates into two rules: Base case and inductive case.
Observation: Difficulty caused by unknown length of list

Let’s write operational semantics for a list of length 2:
Operational Semantics of Variable-Length Expressions

- **Observation**: Difficulty caused by unknown length of list

- Let’s write operational semantics for a list of length 2:

\[ \vdash [i_1 \; i_2] : i_1 + i_2 \]
Operational Semantics of Variable-Length Expressions

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- Base case: List with one integer
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\[ \vdash [i] : i \]
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  \[ \vdash [i] : i \]

- Inductive Case: List with at least two integers
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- **Base case:** List with one integer

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\[ \vdash [R] : i_2 \]
\[ \vdash [i_1, R] : i_1 + i_2 \]

Upshot: To give semantics to variable-length expression, decompose recursively into inductive case(s) and base case(s).

Observe that it is possible to encode computation in this formalism, we will (briefly) see this again towards the end of the class.
Operational Semantics of Variable-Length Expressions

- **Base case:** List with one integer

  \[ \vdash [i] : i \]

- **Inductive Case:** List with at least two integers

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Alternative Semantics

- We can also define the meaning of a list program as follows:

  Base case:

  \[ \vdash i : i \]

  Inductive case:

  \[ \vdash e_1 : i_1 \quad \vdash e_2 : i_2 \]

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  Removing the brackets:

  \[ \vdash e : i \]

  \[ \vdash [e] : i \]
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Removing the brackets:

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\vdash e : i \\
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Are these two semantics equivalent?
Operational Semantics of Application in L

- Last time we only gave operational semantics for the application base case: Two expressions:

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\begin{align*}
E \vdash e_1 : \text{lambda } x. \ e' \\
E \vdash e'_1[e_2/x] : e \\
\hline
E \vdash (e_1 \ e_2) : e
\end{align*}
\]

But the application can have any number of expressions in L. Example: \((x \ y \ z)\) is a valid L expression with meaning \(((x \ y) \ z)\).

Solution: Write inductive case for more than two expressions!
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\begin{align*}
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E \vdash e'_1[e_2/x] : e \\
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Observe: L syntax allows this to indicate associativity and precedence
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**Question:** What is the meaning (operational semantics rule) for \((x)\)?
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- What about an application with one expression, such as \((x)\)?
- This is not an application
- Observe: L syntax allows this to indicate associativity and precedence

- Question: What is the meaning (operational semantics rule) for \((x)\)?
- Answer: 

\[
E \vdash e : e' \\
\frac{}{E \vdash (e) : e'}
\]
List Operations

- Let’s also take a brief look at semantics for some list operations:
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- Consider \(! e\), which evaluated to the head of the list if \( e \) is a list and to \( e \) otherwise
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\]

▶ \( e \) is not a list:

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\frac{E \vdash e : e_1 \ (e_1 \text{ not a list})}{E \vdash \mathtt{!e} : e_1}
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\[
\begin{align*}
E & \vdash e_1 : e'_1 \\
E & \vdash e_2 : e'_2 \text{(} e'_2 \text{ not Nil)} \\
\hline
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List Operations

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\[
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E \vdash e_2 : e_2' (e_2' \text{ not Nil}) \\
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Congratulations!

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- For PA3, you will need to refer to the operational semantics of L in the manual to implement your interpreter.
- The manual is the official source for the semantics of L, not the reference interpreter!
Operational Semantics

- The rules we have written are known as large-step operational semantics.
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- Example: The plus rule

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\begin{align*}
E &\vdash e_1 : i_1 \text{ (integer)} \\
E &\vdash e_2 : i_2 \text{ (integer)} \\
E &\vdash e_1 + e_2 : i_1 + i_2
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- Here, we evaluate both \(e_1\) and \(e_2\) to compute the final value in one (big) step.
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- Alternate formalism for giving semantics: small-step operational semantics.
Small-Step Operational Semantics

- Small-step operational semantics perform only one step of computation per rule invocation.
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- You can think of SSOS as “decomposing” all operations that happen in one rule in LSOS into individual steps
Small Step Operational Semantics

- Small-step operational semantics perform only one step of computation per rule invocation.

- You can think of SSOS as “decomposing” all operations that happen in one rule in LSOS into individual steps.

- This means: Each rule in SSOS has at most one precondition.
SSOS are easiest understood by an example. Consider the integer plus in L written in SSOS:
Small-step Operational Semantics

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- Rule 1: Adding two integers

\[
\langle c_1 + c_2, E \rangle \rightarrow \langle c_1 + c_2, E \rangle
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Small-step Operational Semantics

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\]

- Rule 2: Reducing first expression to an integer

\[
\langle e_1, E \rangle \rightarrow \langle c, E' \rangle
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- **Rule 2: Reducing first expression to an integer**

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  \]

  \[
  \langle e_1 + e_2, E \rangle \rightarrow \langle c + e_2, E' \rangle
  \]

- **Rule 3: Reducing second expression to an integer**

  \[
  \langle e, E \rangle \rightarrow \langle c_2, E' \rangle
  \]

  \[
  \langle c_1 + e, E \rangle \rightarrow \langle c_1 + c_2, E' \rangle
  \]
SSOS in Action

Let’s use these rules to prove what the value of \((2 + 4) + 6\) is:
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\[
\langle (2 + 4) + 6, \_ \rangle \rightarrow \langle 6 + 6, \_ \rangle \rightarrow \langle 12, \_ \rangle
\]
SSOS

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- SSOS are really (conditional) rewrite rules
- The $\beta$ reduction of $\lambda$-calculus is a small-step semantics rule
Recall the large-step operational semantics:

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\begin{align*}
E & \vdash e_1 : \text{lambda } x. \ e'_1 \\
E & \vdash e'_1[e_2/x] : e \\
\hline
E & \vdash (e_1 \ e_2) : e
\end{align*}
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SSOS of the Application

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- What are equivalent SSOS?
SSOS of the Application

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\end{align*}
\]

- What are equivalent SSOS?

\[
\begin{align*}
\langle e'_1[e_2/x], E \rangle & \rightarrow \langle e_3, E' \rangle \\
\langle (\text{lambda } x.e'_1 e_2), E \rangle & \rightarrow \langle e_3, E' \rangle
\end{align*}
\]
Recall the large-step operational semantics, evaluating $e_1$ made a difference:

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\begin{align*}
E \vdash e_1 &: \text{lambda } x. \ e'_1 \\
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E \vdash (e_1 \ e_2) &: e
\end{align*}
\]

What about in SSOS?

For SSOS, other rules will rewrite the expression until it matches the form $\text{lambda } x. \ e'_1$. 
SSOS of the Application

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\end{align*}
$$

▶ What about in SSOS?

▶ For SSOS, other rules will rewrite the expression until it matches the form $\text{lambda } x. \ e'_1$
First try:

\[
\langle e_2, E[x ← e_1] \rangle \rightarrow \langle e_3, \_ \rangle
\]

\[
\langle \text{let } x = e_1 \text{ in } e_2, E \rangle \rightarrow \langle e_3, E \rangle
\]
SSOS of let

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\frac{\langle e_2, E[x \leftarrow e_1]\rangle \rightarrow \langle e_3, -\rangle}{\langle let \ x = e_1 \ in \ e_2, E\rangle \rightarrow \langle e_3, E\rangle}
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- **Notation:** We will write \( \widehat{e} \) to indicate that expression \( e \) has been evaluated as much as possible.
SSOS of let cont.

- Here are the two rules for eager let in SSOS:
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\begin{align*}
\langle e_2, E[x \leftarrow \hat{e}_1] \rangle & \rightarrow \langle e_2, \_ \rangle \\
\langle \text{let } x = \hat{e}_1 \text{ in } e_2, E \rangle & \rightarrow \langle e_3, E \rangle \\
\langle e_1, E \rangle & \rightarrow \langle \hat{e}_1, E' \rangle \\
\langle \text{let } x = e_1 \text{ in } e_2, E \rangle & \rightarrow \langle \text{let } x = \hat{e}_1 \text{ in } e_2, E' \rangle
\end{align*}
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Small-step vs. Big-step Semantics

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- Recall: In BSOS, encountering an undefined expression, such as 3+"duck" got us “stuck”, i.e., we could never satisfy the hypothesis to reach a conclusion.

- In SSOS, undefined expressions also get stuck, i.e. no rule applies.
Advantages of SSOS Cont.

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> **Upshot:** SSOS allow us to distinguish non-termination from errors
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- This allows us to talk about (some) notions of complexity when analyzing small-step semantics.

- Main disadvantage of small step semantics is that they are less intuitive and usually harder to write.

- SSOS also **always** force one order, even if we would like to leave an order undefined.
Conclusion

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- Why: Easier to understand and easier to prove (most) properties with them