Vector and Affine Math I
Points in Euclidean Space $\mathbb{R}^n$

Location in space

Tuple of $n$ coordinates $x, y, z, \text{etc}$

$$p = (p_x, p_y, p_z)$$

Cannot be added or multiplied together
Vectors: “Arrows in Space”

Vectors are **point changes**
Also number tuple: coordinate changes

\[ \vec{v} = (4, 2) \]

\[ \Delta y = 2 \]
\[ \Delta x = 4 \]

Exist **independent** of any reference point
Vector Arithmetic

Subtracting points gives vectors

- Vector between $p$ and $q$: $q - p$

\[ q = (q_x, q_y, q_z) \]

\[ p = (p_x, p_y, p_z) \]
Vector Arithmetic

Subtracting points gives vectors
• Vector between \( p \) and \( q \): \( q - p \)

Add vector to point to get new point
Vector Arithmetic

Vectors can be

- added (tip to tail)
Vector Arithmetic

Vectors can be
• added (tip to tail)
• subtracted
Vector Arithmetic

Vectors can be
- added (tip to tail)
- subtracted
- scaled
Vector Norm

Vectors have magnitude (length or norm)

\[ \| \vec{v} \| = \sqrt{v_x^2 + v_y^2 + v_z^2 + \cdots} \]

- *n*-dimensional Pythagorean theorem
Vector Norm

Vectors have magnitude (length or norm)

\[ \| \vec{v} \| = \sqrt{v_x^2 + v_y^2 + v_z^2 + \cdots} \]

Triangle inequality: \[ \| \vec{v} + \vec{w} \| \leq \| \vec{v} \| + \| \vec{w} \| \]
Unit Vectors

Vectors with $||\vec{v}|| = 1$ unit or normalized

- encode pure direction

Borrowed from physics: “hat notation” $\hat{\vec{v}}$

Any non-zero vector can be normalized:

$$\hat{\vec{v}} = \frac{\vec{v}}{||\vec{v}||}$$
Dot Product

Takes two vectors, returns scalar

\[ \vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z + \cdots \]

• (works in any dimension)

\[ \vec{v} \cdot \hat{w} \text{ is length of } \vec{v} \text{ “in the } \hat{w} \text{ direction”} \]
Dot Product

Takes two vectors, returns scalar

\[ \vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z + \cdots \]

Alternate formula:

\[ \vec{v} \cdot \hat{w} = ||\vec{v}|| \cos \theta \]
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\[ \vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos \theta \]
Projection

Projection onto and out of

\[ w = \frac{u \cdot v}{v \cdot v} \]

\[ u - w \]
Dot Product and Angles

Note $\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$ requires only multiplications and sqrts.

Useful because trig calls are slow.

Also in a pinch (slow): $\theta = \arccos \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$
Cross Product

Takes two vectors, returns vector

\[ \vec{v} \times \vec{w} = (v_y w_z - v_z w_y, v_z w_x - v_x w_z, v_x w_y - v_y w_x) \]

• works **only** in 3D

Direction: perpendicular to both \( \vec{v}, \vec{w} \)

Magnitude: \[ ||\vec{v} \times \vec{w}|| = ||\vec{v}|| ||\vec{w}|| \sin \theta \]
Cross Product Intuition

- Magnitude is area of parallelogram formed by vectors
- Orthogonal direction to vectors
Cross Product Intuition

Use right-hand rule
Cross Product Uses

Easily computes unit vector perpendicular to two given vectors

\[ \hat{n} = \frac{u \times v}{\|u \times v\|} \]

Relation to angles: \( \sin \theta = \frac{\|u \times v\|}{\|u\|\|v\|} \)

Even better: \( \tan \theta = \frac{\|u \times v\|}{u \cdot v} \) no sqrts!
Euclidean Coordinates

A vector in 2D \((v_x, v_y)\) can be interpreted as instructions

“move to the right \(v_x\) and up \(v_y\)”
Euclidean Coordinates

A vector in 2D \((v_x, v_y)\) can be interpreted as instructions

“move to the right \(v_x\) and up \(v_y\)”

In other words: \((v_x, v_y) = v_x \hat{x} + v_y \hat{y}\)
(Finite) Vector Spaces

We say: 2D vectors are vector space of vectors spanned by basis vectors \( \{\hat{x}, \hat{y}\} \)

- basis vectors: “directions” to travel
- span: all linear combinations
Dimension

**Dimension** is size of biggest set of linearly independent basis vectors

Adding all vectors in vector space to point $p$ when dimension is:

- 0: just $p$
- 1: line through $p$
- 2: plane through $p$
- 3+: hyperplane through $p$
Vectors and Bases

- Pick a basis, order the vectors in it
  - All vectors in the space can be represented as a sequence of coordinates

Examples:
- Cartesian 3-space
- Basis \([i \ j \ k]\)
- Linear combination: \(x_i + y_j + z_k\)
- Coordinate representation \([x \ y \ z]\)
Identity Matrix

Square diagonal matrix:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Identity matrix doesn’t change the matrix it’s multiplied by
Inverse Matrix

• For some square matrices, inverse exists:
  \[ AA^{-1} = A^{-1}A = I \]

• A non-invertible matrix is called **singular**
  (determinant is 0)

• Very expensive to compute on large matrices!
Determinant

Maps square matrix to real number

\[
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix} = ad - bc
\]

In 2D measures signed volume of parallelogram
Determinant

In 3D, measures signed volume of parallelepiped

\[
\text{det } \left[ \begin{array}{ccc}
\vec{a}_1^* & \vec{a}_2^* & \vec{a}_3^*
\end{array} \right]
\]
**Transpose**

Flips indices; “reflect about diagonal”

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23}
\end{bmatrix}^T = \begin{bmatrix}
    a_{11} & a_{21} \\
    a_{12} & a_{22} \\
    a_{13} & a_{23}
\end{bmatrix}
\]

Transpose of vector is row vector

\[
\begin{bmatrix}
    v_1 \\
    v_2 \\
    v_3
\end{bmatrix}^T = \begin{bmatrix}
    v_1 & v_2 & v_3
\end{bmatrix}
\]