Physical Simulation
Things We Can Simulate

- Point Masses
- Collision Detection and Response
- Rigid Bodies
- Articulated Systems and Constraints
- Soft Bodies
- Fluid Dynamics
Point Masses

Remember that particle systems are functionally a collection of point masses that obey some set of rules.

What rules might particles in a physical simulation follow?
Newton’s Equations of Motion

Describe motion over time by modeling force in relationship to object trajectory

• $F = ma$

Integrating over time captures a system’s physical behaviors

How to discretize?
Vector Field

At any point in space, function $g(x, t)$ defines a vector field dictating velocity for $x$ at time $t$. 
Particle in a Vector Field

• Particle has a position and a velocity based on the vector field

• How to calculate a new position?
Differential Equations

\[ \dot{x} = g(\vec{x}, t) \]

is a first-order differential equation!
Solve for \( x \) over time by starting at initial point and stepping along the vector field.
Euler’s Method

• Take linear time steps ($\Delta t$) along flow:

$$\vec{x}(t + \Delta t) = \vec{x}(t) + \Delta t \cdot \vec{x}(t) = \vec{x}(t) + \Delta t \cdot g(\vec{x}, t)$$

• Write as a time iteration:

$$\vec{x}^{i+1} = \vec{x}^i + \Delta t \cdot \vec{v}^i$$
Euler across Time Steps

What do you notice about Euler’s method?
Explicit Euler Properties

• Simplest numerical method
• Bigger steps lead to bigger errors
Particle in a Force Field

Now consider a particle with mass in a force field $\mathbf{f}$

We can write out Newton’s law as follows:

$$\vec{f} = m\vec{a} = m\ddot{x}$$

Since $\mathbf{f}$ depends on particle position, velocity and time:

$$\ddot{x} = \frac{\mathbf{f}(x, \dot{x}, t)}{m}$$
Second Order Equations

\[ \ddot{x} = \frac{\vec{f}(\vec{x}, \dot{x}, t)}{m} \]

is a second order differential equation…
we’d rather not deal with this!

Rather than solve directly, create a pair of coupled first order equations:

\[
\begin{bmatrix}
\dot{x} = \dot{v} \\
\dot{v} = \frac{\vec{f}(\vec{x}, \dot{v}, t)}{m}
\end{bmatrix}
\]
Differential Equation Solver

Since
\[
\begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
v \\
f/m
\end{bmatrix}
\]

Euler’s method:
\[
\ddot{x}(t + \Delta t) = \ddot{x}(t) + \dot{x}(t) \Delta t
\]
\[
\dot{x}(t + \Delta t) = \dot{x}(t) + \ddot{x}(t) \Delta t
\]

With substitutions:
\[
\ddot{x}(t + \Delta t) = \ddot{x}(t) + \frac{\vec{f}(\vec{x}, \dot{x}, t)}{m} \Delta t
\]
\[
\dot{x}(t + \Delta t) = \dot{x}(t) + \frac{\vec{f}(\vec{x}, \dot{x}, t)}{m} \Delta t
\]
Euler Iterative Form

\[ x^{i+1} = x^i + v^i \Delta t \]

\[ v^{i+1} = v^i + \frac{f^i}{m} \Delta t \]

Still performs poorly for large time steps!

Ideally we want a more stable integrator…
Verlet Integration

A better solver with greater stability and no additional computational overhead

Three versions:

• Position
• Velocity
• Leapfrog
Position Verlet

Handles velocities implicitly:

\[ \vec{x}^{i+1} = \vec{x}^i + (\vec{x}^i - \vec{x}^{i-1}) + \dot{\vec{v}} \Delta t^2 \]

\[ \vec{x}^{i-1} = \vec{x}^i \]

- Requires constant time steps and two steps to start
- Commonly used in games
Applying Forces

Each particle experiences a force/forces

Common forces:

• Constant (gravity)
• Position/time dependent (force fields)
• Velocity dependent (drag)
• Combinations (damped springs)
Force Examples

Gravity: \[ \vec{f}_{grav} = m\vec{G} \]

Viscous drag: \[ \vec{f}_{drag} = -k\vec{v} \]

One body spring: \[ \vec{f} = -k_{spring}(|\Delta \vec{x}| - r) \]

One body damped spring:
\[ \vec{f} = -[k_{spring}(|\Delta \vec{x}| - r) + k_{damp} |\vec{v}|] \]
Collision Detection and Response

Collision Detection: Determine when an intersection has happened

Collision Response: Determine what to do when intersection detected
Collision Detection

A very familiar problem!  
(Think ray-tracing)

Must also consider particle velocity
Collision Response

• After Contact (a posteriori)
  • Run simulation
  • “Roll back” if intersection occurs
• Before Contact (a priori)
  • Predict time of collision
  • Update position accordingly
• Resting Contact
  • Two objects are in contact with each other
  • A surprisingly difficult special case!
Rigid Bodies

Extends idea of point-mass

• Bodies can be interconnected
• Bodies are rigid relative to each other
Articulated Systems and Constraints

Not all rotations are physically plausible
Solve by limiting joint movement with constraints
Use of inverse kinematics to solve for all joint angles based on final position of child bones
Soft Bodies

Distance between particles is not fixed
Generally a very expensive computation
Easier to simulate as a system of rigid body springs
Fluid Dynamics

Describes the flow of fluids and gases

Models properties such as:

• Flow velocity
• Pressure
• Density
• Temperature
Navier-Stokes

Applies Newton’s second law to fluid motion to calculate flow velocity

Requires solving for fluid’s diffusion (change in concentration) and advection (transport of material)

Can solve using grid-based or particle-based methods
Grid vs Particle

(a) Grid representation

(b) Particle representation
Curl-Noise

Non-physically-based method for approximating fluid flow

• Create a vector field using Perlin noise
• Take curl (rotation) of this field to generate a divergence-free (doesn’t shrink or expand) velocity field

Curl Noise Demo

https://www.youtube.com/watch?v=8TNZS2AkFNs