CS311: Discrete Math for Computer Science, Spring 2015

Additional Exercises, with Solutions

1. For each formula, determine whether it is true. If it is false then find a counterexample.

(a) $\forall n \left(n > \sqrt{2} \rightarrow n > \sqrt{3}\right)$.

Answer: true. Indeed, if $n > \sqrt{2}$ then n is at least 2, so that $n > \sqrt{3}$.

(b) $\forall n(n = 10^2 + 11^2 + 12^2 \leftrightarrow n = 13^2 + 14^2)$.

Answer: true, because $10^2 + 11^2 + 12^2$ and $13^2 + 14^2$ both equal 365.

(c) $\forall xy(x > 2 \land y > 2 \rightarrow xy > 8)$.

Answer: false; counterexample: $x = 2.5$, $y = 3$.

2. Express in logical notation:

The sum of any two positive integers is greater than 1.

Answer: $\forall mn(m > 0 \land n > 0 \rightarrow m + n > 1)$.

3. For each formula, either prove it by exhaustion or find a counterexample.

(a) $\forall n(3 \leq n \leq 5 \rightarrow 2^n \geq n + 5)$.

Proof: the inequality $2^n \geq n + 5$ holds for all values of n satisfying the antecedent (3, 4, and 5).

(b) $\forall n(3 \leq n \leq 5 \rightarrow 2^n \leq n + 25)$.

Answer: false; counterexample: $n = 5$.

4. For each formula, determine which of its two variables is free. Determine whether the formula is true when the value of the free variable is 0. If it is true then find a witness.

(a) $\exists n(x > 2^n)$.

Solution: x is free. When x is 0, the formula becomes $\exists n(0 > 2^n)$, and this formula is false.

(b) $\exists x(x > 2^n)$.

Solution: n is free. When n is 0, the formula becomes $\exists x(x > 1)$, and this formula is true. Witness: $x = 2$.

5. Rewrite the expression

$$\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}} + \cdots + \frac{k-1}{\sqrt{k}}$$

in Sigma-notation. *Answer:*

$$\sum_{i=1}^{k-1} \frac{i}{\sqrt{i+1}}.$$

6. For every positive integer n, the number A_n is defined by the formula

$$A_n = \sum_{i=n}^{2n} i.$$

(a) Find A_1. *Solution:*

$$A_1 = \sum_{i=1}^{2} i = 1 + 2 = 3.$$

(b) Find $A_{100} - A_{99}$. *Solution:*

$$\begin{aligned} A_{100} - A_{99} = (100 + 101 + \cdots + 199 + 200) - (99 + 100 + \cdots + 197 + 198) \\ = 199 + 200 - 99 = 300. \end{aligned}$$