CS311: Discrete Math for Computer Science, Spring 2015

Additional Exercises, with Solutions

1. The sequence $X_1, X_2, \dots$ is defined by the formulas

$$X_n = \begin{cases} 3n, & \text{if } n \text{ is even,} \\ 4n, & \text{otherwise.} \end{cases}$$

(a) Find the first 3 members of this sequence.

Answer: $X_1 = 4$, $X_2 = 6$, $X_3 = 12$.

(b) Prove that all members of this sequence are even.

Solution: Consider two cases.
Case 1: n is odd. Then $X_n = 4n = 2(2n)$.
Case 2: n is even. Then $n = 2k$ for some k. So $X_n = 3n = 3(2k) = 2(3k)$.

(c) Prove that all members of this sequence are greater than 3.

Solution: Consider two cases.
Case 1: n is odd. Then n is at least 1. Now $X_n = 4n \geq 4 \cdot 1 > 3$.
Case 2: n is even. Then n is at least 2. Now $X_n = 3n \geq 3 \cdot 2 > 3$.

(d) We conjecture that there exist coefficients a and b such that the sequence X_n satisfies the condition

$$X_n = an + bn(-1)^n$$

for all positive integers n. Find a pair of numbers a, b that may satisfy this condition.

Solution: For convenience, define $f(n) = an + bn(-1)^n$. Then $f(1) = a - b$ and $f(2) = 2a + 2b$. So if $f(n) = X_n$, then we at least know that

$$\begin{aligned} a - b &= 4 \\ 2a + 2b &= 6. \end{aligned}$$

Solving gives us $a = \frac{7}{2}$ and $b = -\frac{1}{2}$.

(e) Check that for the numbers that you found this condition is indeed satisfied for all n.

Solution: Consider two cases.
Case 1: n is odd. Then $X_n = 4n$. Also $(-1)^n = -1$, so $f(n) = \frac{7}{2}n + \frac{1}{2}n = 4n$.
Case 2: n is even. Then $X_n = 3n$. Also $(-1)^n = 1$, so $f(n) = \frac{7}{2}n - \frac{1}{2}n = 3n$.

2. Prove that for every nonnegative integer n,

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3};$$

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Solution:

Proof of the first formula by induction:

Basis: $n = 0$. The given formula turns into the correct equality $1^2 = \frac{1 \cdot 1 \cdot 3}{3}$. *Induction step:* Assuming that the given formula is true for n, we can prove that

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 + (2n+3)^2 = \frac{(n+2)(2n+3)(2n+5)}{3}$$

as follows:

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 + (2n+3)^2 &= \frac{(n+1)(2n+1)(2n+3)}{3} + (2n+3)^2 \\
&= (2n+3)\left(\frac{(n+1)(2n+1)}{3} + (2n+3)\right) \\
&= (2n+3)\frac{2n^2+3n+1+6n+9}{3} \\
&= (2n+3)\frac{2n^2+9n+10}{3} \\
&= (2n+3)\frac{(n+2)(2n+5)}{3} \\
&= \frac{(n+2)(2n+3)(2n+5)}{3}.
\end{aligned}$$

Proof of the second formula by induction:

Basis: $n = 0$. The given formula turns into the correct equality $0 = \frac{0 \cdot 1 \cdot 2}{3}$. *Induction step:* Assuming that the given formula is true for n, we can prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

as follows:

$$\begin{aligned}
1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2) &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\
&= (n+1)(n+2)\left(\frac{n}{3} + 1\right) \\
&= (n+1)(n+2) \cdot \frac{n+3}{3} \\
&= \frac{(n+1)(n+2)(n+3)}{3}.
\end{aligned}$$

3. Prove that for every positive integer n,

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}.$$

Solution: We will prove the formula by induction. *Basis:* $n = 1$. The given formula turns into the correct equality $1^2 = (-1)^0 \cdot \frac{1 \cdot 2}{2}$. *Induction step:* Assuming that the given formula is true for n, we can prove that

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 + (-1)^n(n+1)^2 = (-1)^n \frac{(n+1)(n+2)}{2}$$

as follows:

$$\begin{aligned} 1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 + (-1)^n(n+1)^2 &= (-1)^{n-1}\frac{n(n+1)}{2} + (-1)^n(n+1)^2 \\ &= (-1)^{n-1}(n+1)\left(\frac{n}{2} - (n+1)\right) \\ &= (-1)^{n-1}(n+1)\frac{-n-2}{2} \\ &= (-1)^n \frac{(n+1)(n+2)}{2}. \end{aligned}$$

4. Determine which positive integers n satisfy the inequality $3^n > 2^{n+2}$. Prove that your answer is correct.

Solution. This inequality holds once $n \geq 4$ but not before that. Proof:

The claim that the inequality $3^n > 2^{n+2}$ is false for $1 \leq n \leq 3$ is proved by the following table:

n	3^n	2^{n+2}
1	3	8
2	9	16
3	27	32

To see that this inequality holds for $n \geq 4$, we will use induction. *Basis:* $n = 4$. Then $3^n = 3^4 = 81 > 64 = 2^{4+2} = 2^{n+2}$. *Inductive step:* Suppose $3^n > 2^{n+2}$ for some $n \geq 4$. Then multiplying by 3 gives us $3^n \cdot 3 > 2^{n+2} \cdot 3$. So

$$3^{n+1} = 3^n \cdot 3 > 2^{n+2} \cdot 3 > 2^{n+2} \cdot 2 = 2^{n+3}.$$

5. This question is about the sequence Y_n, defined in Part 4 of our lecture notes. Is it true that for every even n, Y_n is even? Prove that your answer is correct.

Solution: Yes, this is true. Proof: Suppose that n is even. Then $n = 2k$ for some k. So $Y_n = 2Y_{n-1} + n = 2(Y_{n-1} + k)$.