CS311: Discrete Math for Computer Science, Spring 2015

Homework Assignment 5, with Solutions

1. Prove that for every nonnegative integer n

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Solution: We prove that the statement above is true using induction on n. *Basis:* If $n = 0$, we observe that $\sum_{i=1}^{n} i^3 = 0$ and $\frac{n^2(n+1)^2}{4} = \frac{0 \cdot 1}{4} = 0$. So, the statement is true when $n = 0$. *Induction step:* Assume that the given statement is true for some n. We can prove that

$$\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4}$$

as follows:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 && \text{Using the induction hypothesis} \\
&= (n+1)^2 \left(\frac{n^2}{4} + (n+1) \right) \\
&= (n+1)^2 \cdot \frac{n^2 + 4n + 4}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}$$

2. Prove that $n! > 3^n$ if n is an integer greater than 6.

Solution: We will prove this statement using induction. *Basis:* If $n = 7$, we observe that $n! = 7! = 5040$ and $3^n = 3^7 = 2187$. Since $5040 > 2187$, we conclude that $n! > 3^n$ when $n = 7$. *Induction step:* Assume that $n! > 3^n$ for some $n \geq 7$. We can prove that $(n+1)! > 3^{n+1}$ as follows:

$$\begin{aligned}
(n+1)! &= (n+1) \cdot n! && \text{Using the recursive definition of !} \\
&> (n+1) \cdot 3^n && \text{Using the induction hypothesis} \\
&> 3 \cdot 3^n && \text{Since } n+1 > 3 \text{ when } n > 6 \\
&= 3^{n+1}.
\end{aligned}$$

3. Find all nonnegative integers n such that $3^n > 4n!$. Prove that your answer is correct.

Solution: We begin by tabulating the values of 3^n and $4n!$ for small non-negative values of n.

n	3^n	$4n!$
0	1	4
1	3	4
2	9	8
3	27	24
4	81	96
5	243	480
6	729	4320

It appears that the inequality $3^n > 4n!$ holds for only two values of n: $n = 2$ and $n = 3$.

To prove that this answer is correct, we need to establish the following claim:

For all values of n that are greater than 3, $3^n \leq 4n!$.

We will prove it by induction, *Basis:* $n = 4$. Then $3^n = 81$ and $4n! = 96$, so the given inequality holds. *Induction step:* Assume that for some value of n such that $n \geq 4$, $3^n \leq 4n!$. We can prove that $3^{n+1} \leq 4(n+1)!$ as follows:

$$
\begin{aligned}
3^{n+1} &= 3 \cdot 3^n \\
&\leq 3 \cdot 4n! && \text{Using the induction hypothesis} \\
&\leq (n+1) \cdot 4n! && \text{Since } 3 \leq n+1 \\
&= 4(n+1)! && \text{Since } (n+1) \cdot n! = (n+1)!
\end{aligned}
$$

4. Using the recursive definition of $n!$ in Part 4 of Lecture Notes, calculate 4! in two ways: by eager evaluation and by lazy evaluation.

Solution.

Eager evaluation:

$$
\begin{aligned}
1! &= 1 \cdot 0! = 1 \cdot 1 = 1. \\
2! &= 2 \cdot 1! = 2 \cdot 1 = 2. \\
3! &= 3 \cdot 2! = 3 \cdot 2 = 6. \\
4! &= 4 \cdot 3! = 4 \cdot 6 = 24.
\end{aligned}
$$

Lazy evaluation:

$$
\begin{aligned}
4! &= 4 \cdot 3! \\
&= 4 \cdot 3 \cdot 2! = 12 \cdot 2! \\
&= 12 \cdot 2 \cdot 1! = 24 \cdot 1! \\
&= 24 \cdot 1 \cdot 0! = 24 \cdot 0! \\
&= 24 \cdot 1 = 24.
\end{aligned}
$$

5. The numbers $U_0, U_1, U_2, \ldots$ are defined by the formula

$$U_n = \sum_{i=1}^{n} (2i-1)^2.$$

Define this sequence using recursion, instead of sigma-notation.

Answer:

$$\begin{aligned} U_0 &= 0, \\ U_{n+1} &= U_n + (2n+1)^2. \end{aligned}$$

6. The numbers $V_0, V_1, V_2, \ldots$ are defined by the formulas

$$\begin{aligned} V_0 &= 10, \\ V_{n+1} &= V_n(n^2 + n - 90) + 1. \end{aligned}$$

Find V_1 and V_{11} without a calculator.

Solution:

$$V_1 = V_0 \cdot (0^2 + 0 - 90) + 1 = 10 \cdot (-90) + 1 = -899;$$

$$\begin{aligned} V_{11} &= V_{10} \cdot (10^2 + 10 - 90) + 1 \\ &= 20 \cdot V_{10} + 1 \\ &= 20 \cdot (V_9 \cdot (9^2 + 9 - 90) + 1) + 1 \\ &= 20 \cdot (V_9 \cdot 0 + 1) + 1 \\ &= 20 \cdot 1 + 1 \\ &= 21. \end{aligned}$$

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