CS311: Discrete Math for Computer Science, Spring 2015

Homework Assignment 7, with Solutions

1. Without a calculator, determine which of the numbers

$$10^{30},\ 10^{50},\ 10^{70}$$

gives the best approximation to the value of the fraction $\dfrac{100!}{(50!)^2}$. Justify your answer.

Solution. Using the approximation

$$n! \approx \left(\frac{n}{e}\right)^n$$

we calculate:

$$\frac{100!}{(50!)^2} \approx \frac{\left(\frac{100}{e}\right)^{100}}{\left(\left(\frac{50}{e}\right)^{50}\right)^2} = \frac{\left(\frac{100}{e}\right)^{100}}{\left(\frac{50}{e}\right)^{100}} = 2^{100} = \left(2^{10}\right)^{10} \approx 1000^{10} = \left(10^3\right)^{10} = 10^{30}.$$

2. Prove that

(i) $n^2 + n + 1 = O(n^2)$,

(ii) $3 \cdot 2^n + 100 = O(2^n)$,

(iii) $e^n + e^{n+1} = O(e^n)$.

Solution.

(i) Take $C = 3$ and $N = 1$. We now claim that for all $n \geq N$,

$$n^2 + n + 1 \leq 3n^2.$$

Proof:

$$\begin{aligned} n^2 + n + 1 &\leq n^2 + n^2 + 1 && \text{Since } n \leq n^2 \text{ when } n \geq 1 \\ &\leq n^2 + n^2 + n^2 && \text{Since } 1 \leq n^2 \text{ when } n \geq 1 \\ &= 3n^2. \end{aligned}$$

(ii) Take $C = 103$ and $N = 1$. We now claim that for all $n \geq N$,

$$3 \cdot 2^n \leq 103 \cdot 2^n.$$

Proof:

$$\begin{aligned} 3 \cdot 2^n + 100 &\leq 3 \cdot 2^n + 100 \cdot 2^n && \text{Since } 2^n > 1 \text{ when } n \geq 1 \\ &= 103 \cdot 2^n \end{aligned}$$

(iii) Take $C = 4$ and $N = 1$. We now claim that for all $n \geq N$,

$$e^n + e^{n+1} \leq 4 \cdot e^n.$$

Proof:

$$\begin{aligned} e^n + e^{n+1} &= (e+1) \cdot e^n \\ &\leq 4 \cdot e^n && \text{Since } e + 1 \leq 4 \end{aligned}$$

3. Let A be the set $\{\{1\}, \{2\}, \{3\}\}$.

(i) How many elements does A have?

(ii) Does A have a pair of different elements x, y such that $x \subseteq y$?

(iii) How many subsets does A have?

(iv) Does A have a pair of different subsets x, y such that $x \subseteq y$?

Justify your answers.

Solution.

(i) Set A has 3 elements: $\{1\}$, $\{2\}$, and $\{3\}$.

(ii) No. Neither element of A is contained in another element of A.

(iii) A set containing n elements has 2^n subsets. Since A has 3 elements, it has 8 subsets.

(iv) Yes. For example, $\{\{1\}\} \subseteq \{\{1\}, \{2\}\}$.