Lecture Notes:
Discrete Mathematics for Computer Science

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Part 1. Logical Notation

Propositional Connectives and Quantifiers

Logical notation uses symbols of two kinds: *propositional connectives*, such as

$$\wedge \text{ (``and''}), \quad \vee \text{ (``or''}), \quad \neg \text{ (``not''}),$$

and *quantifiers*

$$\forall \text{ (``for all''}), \quad \exists \text{ (``there exists''}).$$

The symbol $\wedge$, called *conjunction*, and the symbol $\vee$, called *disjunction*, are *binary* connectives, because each of them is used to form a compound proposition from *two* propositions. The negation symbol $\neg$ is a *unary* connective.

The symbol $\forall$ is called the *universal* quantifier; the symbol $\exists$ is the *existential* quantifier.

Examples of Logical Formulas

> In this class, we use x, y, z as variables for real numbers,
> and i, j, k, l, m, n as variables for integers.

$$
\begin{array}{rll}
(x > 5) \wedge (x < 6) & \text{has the same meaning as} & 5 < x < 6; \\
(2x > 1) \vee (2x < 1) & \text{has the same meaning as} & 2x \neq 1; \\
\neg(x > 1) & \text{has the same meaning as} & x \leq 1.
\end{array}
$$

The logical formula

$$\forall x(x^2 + 2x + 2 = (x + 1)^2 + 1)$$

expresses the assertion that the equality $x^2 + 2x + 2 = (x + 1)^2 + 1$ holds for all real numbers x.

The logical formula

$$\exists x(x^2 + 2x + 2 = 0)$$

expresses the (incorrect) assertion that the equation $x^2 + 2x + 2 = 0$ has at least one solution.

The assertion "there exists a negative number x such that its square is 2" can be written as

$$\exists x(x < 0 \wedge x^2 = 2).$$

(Do not write $\exists x < 0(x^2 = 2)$; this is not considered a valid logical formula.)

Truth Tables

The *truth tables* for propositional connectives show how we can determine whether a compound proposition is true if we know which of its component propositions are true. Here is the truth table for conjunction and disjunction:

p	q	$p \wedge q$	$p \vee q$
F	F	F	F
F	T	F	T
T	F	F	T
T	T	T	T

The symbols T and F here stand for *true* and *false*. They are called *truth values*.

The truth table for negation looks like this:

p	$\neg p$
F	T
T	F

Witnesses and Counterexamples

How can we argue that a logical formula beginning with $\exists x$ is true? This can be done by specifying a value of x that satisfies the given condition. Such a value is called a *witness*. For instance, to argue that the assertion

$$\exists x(4 < x^2 < 5)$$

is true, we can use the value $x = 2.1$ as a witness.

How can we argue that a logical formula beginning with $\forall x$ is false? This can be done by specifying a value of x that does *not* satisfy the given condition. Such a value is called a *counterexample*. For instance, to argue that the assertion

$$\forall x(x^2 \geq x) \tag{1}$$

is false, we can use the value $x = \frac{1}{2}$ as a counterexample.

The assertion that formula (1) is false can be expressed in two ways: by the negation of formula (1):

$$\neg \forall x(x^2 \geq x)$$

and also by saying that there exists a counterexample:

$$\exists x \neg (x^2 \geq x).$$

This is an instance of a general fact: the combinations of symbols

$$\neg \forall x \quad \text{and} \quad \exists x \neg$$

have the same meaning.

In the following example we use the notation $m \mid n$ to express that the integer m divides the integer n (or, in other words, that n is a multiple of m). We want to show that the assertion

$$\forall n(2 \mid n \vee 3 \mid n)$$

("every integer is a multiple of 2 or a multiple of 3") is false. What value of n can be used as a counterexample? We want the formula $2 \mid n \vee 3 \mid n$ to be false. According to the truth table for disjunction, this formula is false only when both $2 \mid n$ and $3 \mid n$ are false. In other words, n should be neither a multiple of 2 nor a multiple of 3. The simplest counterexample is $n = 1$.

The formula

$$\exists mn(m^2 + n^2 = 10) \tag{2}$$

expresses that 10 can be represented as the sum of two complete squares. It is different from the formulas with quantifiers that we have seen before in that the existential quantifier is followed here by two variables, not one. To give a witness justifying this claim, we need to find a pair of values m, n such that $m^2 + n^2 = 10$. For instance, the pair of values $m = 3$, $n = 1$ provides a witness.

Note that when we write "can be represented" in logical notation, the existential quantifier is used.

Implication

The binary propositional connective $\rightarrow$ is called *implication*. It represents the combination "if ... then." For instance, the logical formula

$$\forall n(4 \mid n \rightarrow 2 \mid n) \tag{3}$$

says: for all n, if n is a multiple of 4 then n is even. The assertion "the cube of any positive number is positive also" can be written as

$$\forall x(x > 0 \rightarrow x^3 > 0). \tag{4}$$

(Do not write $\forall x > 0(x^3 > 0)$; this is not considered a valid logical formula.) The formula to the left of $\rightarrow$ (in this example, $x > 0$) is called the *antecedent*. The formula to the right of $\rightarrow$ (in this example, $x^3 > 0$) is called the *consequent*.

In the truth table below, the column for implication is added to the columns for conjunction and disjunction:

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$
F	F	F	F	T
F	T	F	T	T
T	F	F	T	F
T	T	T	T	T

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You see that an implication is false in only one case: when its antecedent is true but its consequent is false.

For example, consider the formula

$$\forall x(x > 5 \rightarrow 2x > 20).$$

This formula is false, because we can find a number x such that the antecedent $x > 5$ is true but the consequent $2x > 20$ is false. For instance, $x = 6$ can be used as a counterexample.

Consider the formula

$$\forall xy(x > 0 \wedge y < 0 \rightarrow x + y = 0). \tag{5}$$

This formula is false, because we can find a pair of numbers x, y such that the antecedent $x > 0 \wedge y < 0$ is true but the consequent $x + y = 0$ is false. According to the truth table for conjunction, to make the antecedent true we should make each of the formulas $x > 0$, $y < 0$ true. In other words, to give a counterexample we need to find a pair of numbers x, y such that x is positive, y is negative, and $x + y$ is different from 0. For instance, the pair $x = 1$, $y = -2$ is a counterexample.

The *converse* of an implication is obtained by swapping its antecedent with its consequent. The converse of a true implication may be true or may be false. For instance, the converse of (3) is

$$\forall n(2 \mid n \rightarrow 4 \mid n);$$

this formula is false. (The value $n = 2$ can be used as a counterexample.) The converse of (4) is

$$\forall x(x^3 > 0 \rightarrow x > 0);$$

this formula is true.

Equivalence

The binary propositional connective $\leftrightarrow$ is called *equivalence*. It represents the combination "if and only if" (sometimes abbreviated as "iff"). For instance, the logical formula

$$\forall n(6 \mid n \leftrightarrow 2 \mid n \wedge 3 \mid n) \tag{6}$$

expresses that an integer is a multiple of 6 iff it is both a multiple of 2 and a multiple of 3.

The last column in the truth table below shows that an equivalence is true whenever its left-hand side and its right-hand side have the same truth value.

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

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The equivalence $p \leftrightarrow q$ has the same meaning as the conjunction of two implications

$$(p \rightarrow q) \wedge (q \rightarrow p).$$

For instance, the assertion "x^3 is positive iff x is positive" can be written either as an equivalence

$$\forall x(x^3 > 0 \leftrightarrow x > 0)$$

or as the conjunction of two implications:

$$\forall x((x^3 > 0 \rightarrow x > 0) \wedge (x > 0 \rightarrow x^3 > 0)).$$

To provide a counterexample for an equivalence, we need to make one of its two sides true while the other is false. For instance, the formula

$$\forall x(x^2 > 0 \leftrightarrow x > 0)$$

is false, and $x = -1$ can be used as a counterexample: for this value of x, the left-hand side is true, and the right-hand side is false.

Proofs by Exhaustion

The implication

$$\forall n(-1 \leq n \leq 1 \rightarrow n^3 = n)$$

is easy to prove, because only three values of n satisfy its antecedent $-1 \leq n \leq 1$, and we can check for each of them individually that it satisfies the consequent $n^3 = n$ as well. This kind of reasoning is called "proof by exhaustion." It is not applicable if the antecedent is satisfied for infinitely many values of variables. For instance, the implication

$$\forall x(-1 \leq x \leq 1 \rightarrow -1 \leq x^3 \leq 1)$$

cannot be proved by exhaustion, because there are infinitely many real numbers between -1 and 1.

Free and Bound Variables

When a formula begins with $\forall x$ or $\exists x$, we say that the variable x is *bound* in it. If a quantifier is followed by several variables then all of them are bound. When a variable is not bound then we say that it is *free* in the formula. For instance, in the formula

$$\exists i(j = i^2) \tag{7}$$

the variable i is bound, and the variable j is free. In the formula

$$\exists ij(i^2 + j^2 = 29) \tag{8}$$

both i and j are bound.

The difference between free and bound variables is important because the truth value of a formula depends on the values of its free variables, but does not depend on the values of its bound variables. For instance, formula (7) expresses that j is a complete square; whether or not it is true depends on the value of its free variable j. Formula (8) expresses that 29 can be represented as the sum of two squares; we don't need to specify the values of any variables before we ask whether this formula is true.

Replacing a bound variable by a new variable does not change the meaning of a formula. For instance, the formula $\exists k(j = k^2)$ has the same meaning as (7): it says that j is a complete square.

Digression: Bound Variables in Algebra and Calculus

Quantifiers are not the only mathematical symbols that bind variables. Sigma notation for sums of numbers creates bound variables as well. For instance, the sum of the squares of the numbers from 1 to 4 can be written as

$$\sum_{n=1}^{4} n^2.$$

Here n is a bound variable. The value of this expression is 30; it does not depend on n. Replacing n by a different variable does not change the meaning of the expression. For instance, the value of the expression

$$\sum_{i=1}^{4} i^2$$

is 30 as well.

The expression

$$\sum_{i=1}^{n} i^2$$

denotes the sum of the squares of the numbers from 1 to n. In this expression, n is free, and i is bound. The value of this expression depends on n, but not on i.

In the expression

$$\lim_{n \to \infty} \frac{2n+1}{n+1}$$

the variable n is bound. The value of the limit is 2; it does not depend on n.

In the definite integral

$$\int_0^1 x^2 dx$$

the variable x is bound. The value of the integral is $\frac{1}{3}$; it does not depend on x. In the expression

$$\int_0^x t^2 dt$$

t is bound and x is free. The value of the integral is $\frac{x^3}{3}$; it depends on x, but not on t.

Nested Quantifiers

Some logical formulas include "nested" quantifiers, where one quantifier is within the scope of another:

$$\exists m \exists n (m^2 + n^2 = 10), \tag{9}$$

$$\forall x \forall y (x > 0 \wedge y < 0 \rightarrow x + y = 0), \tag{10}$$

$$\forall x \exists y (y > x). \tag{11}$$

If both quantifiers are existential, as in (9), or both are universal, as in (10), then the second quantifier can be dropped, and the formula will not change its meaning. For instance, formula (9) has the same meaning as (2), and (10) has the same meaning as (5). But using a combination of a universal quantifier with an existential quantifier, as in (11), we can express more complex ideas than with just one quantifier. For instance, formula (11) says that for every real number x we can find a real number y that is greater than x. This formula is true: we can take $y = x + 1$ as a witness. This equality shows how to calculate a witness y for any given value of x.

Note that the order of quantifiers in (11) is essential. The formula

$$\exists y \forall x (y > x),$$

which differs from (11) by the order of quantifiers, says that there exists a *single value* of y that is greater than *all* real numbers. This formula is false.

On the other hand, the formula

$$\exists y \forall x (y > \sin x)$$

is true. Indeed, $y = 2$ can be taken as a witness.