Lecture Notes:
Discrete Mathematics for Computer Science

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Part 2. Definitions and Proofs by Cases

Defining a Function by Cases

Functions in algebra are usually defined by formulas, for instance:

$$f(x) = x^2 + x + 1.$$

Sometimes a function is defined by several formulas corresponding to different values of the argument, as in these examples:

$$g(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$h(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{otherwise;} \end{cases}$$

$$\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Such a definition can be rewritten in logical notation so that each case will be represented by an implication:

$$\forall x((x \geq 0 \rightarrow g(x) = x^2) \wedge (x < 0 \rightarrow g(x) = 0));$$

$$\forall x((x \geq 0 \rightarrow h(x) = 1) \wedge (x < 0 \rightarrow h(x) = -1));$$

$$\forall x((x \geq 0 \rightarrow |x| = x) \wedge (x < 0 \rightarrow |x| = -x));$$

$$\forall x((x > 0 \rightarrow \text{sgn}(x) = 1) \wedge (x = 0 \rightarrow \text{sgn}(x) = 0) \wedge (x < 0 \rightarrow \text{sgn}(x) = -1)).$$

Defining a Sequence by Cases

A definition by cases can be used also to describe a sequence of numbers. For instance, the sequence

n	1	2	3	4	5	6	...
A_n	2	5	2	5	2	5	...

can be defined by the formulas

$$A_n = \begin{cases} 2, & \text{if } n \text{ is odd,} \\ 5, & \text{otherwise.} \end{cases}$$

In logical notation:

$$\forall n((2 \nmid n \to A_n = 2) \land (2 \mid n \to A_n = 5)).$$

The sequence of powers of -1 can be described by a similar definition:

$$(-1)^n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ 1, & \text{otherwise.} \end{cases}$$

Digression: The Method of Undetermined Coefficients

It turns out that the sequence A_n defined above can be characterized by a single formula that works for all values of n, both odd and even. This formula has the form

$$A_n = a(-1)^n + b, \tag{1}$$

where a and b are numerical coefficients. We can find the values of a and b by *the method of undetermined coefficients*, which is widely used in mathematics. Since we want formula (1) to hold for all positive integers n, it should hold, in particular, for $n = 1$ and for $n = 2$. For $n = 1$ this formula gives

$$2 = -a + b.$$

For $n = 2$ we get

$$5 = a + b.$$

We can solve these equations for a and b and find:

$$a = \frac{3}{2},\ b = \frac{7}{2}.$$

If we replace the letters a and b in formula (1) with these numerical values, we'll get:

$$A_n = \frac{3}{2}(-1)^n + \frac{7}{2}. \tag{2}$$

Proofs by Cases

Now we want to check that formula (2) holds for all values of n, and not only for 1 and 2. Consider two cases. *Case 1:* n is odd. Then

$$A_n = 2; \quad \frac{3}{2}(-1)^n + \frac{7}{2} = \frac{3}{2}(-1) + \frac{7}{2} = -\frac{3}{2} + \frac{7}{2} = 2.$$

Case 2: n is even. Then

$$A_n = 5; \quad \frac{3}{2}(-1)^n + \frac{7}{2} = \frac{3}{2} \cdot 1 + \frac{7}{2} = \frac{3}{2} + \frac{7}{2} = 5.$$

So formula (2) holds in both cases.

This proof of formula (2) is a *proof by cases.* Such proofs are often used in mathematics. Its structure can be symbolically represented by this "inference rule":

$$\frac{F \rightarrow G \qquad \neg F \rightarrow G}{G}.$$

This figure tells us that if we established the two *premises* $F \rightarrow G$, $\neg F \rightarrow G$ then we can derive the *conclusion* G. The first premise says that G true if F is true (Case 1); the second premise says that G true if F is false (Case 2). In the example above, the condition F that distinguishes Case 1 from Case 2 is "n is odd" (symbolically, $2 \not| n$), and the conclusion G is formula (2).

As another example of proof by cases, we will show that for every real number x,

$$|x|^2 = x^2.$$

Recall that $|x|$ is defined by the formulas

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{otherwise.} \end{cases}$$

Case 1: $x \geq 0$. Then $|x| = x$, so that $|x|^2 = x^2$. *Case 2:* $x < 0$. Then $|x| = -x$, so that $|x|^2 = (-x)^2 = x^2$.

Distinguishing Between Three Cases

In some proofs we distinguish between several cases, not just two. Let us prove, for instance, that for every real number x,

$$\text{sgn}(x) \cdot |x| = x.$$

Recall that $\text{sgn}(x)$ is defined by the formulas

$$\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Case 1: $x < 0$. Then
$$\mathrm{sgn}(x) \cdot |x| = (-1) \cdot (-x) = x.$$
Case 2: $x = 0$. Then
$$\mathrm{sgn}(x) \cdot |x| = 0 \cdot x = 0 = x.$$
Case 3: $x > 0$. Then
$$\mathrm{sgn}(x) \cdot |x| = 1 \cdot x = x.$$

The structure of this proof can be represented by an inference rule with four premises:

$$\frac{F_1 \vee F_2 \vee F_3 \qquad F_1 \to G \qquad F_2 \to G \qquad F_3 \to G}{G}.$$

The first premise expresses that the conditions F_1, F_2, F_3, which describe the three cases, cover all possibilities. In the example above, this premise is

$$(x < 0) \vee (x = 0) \vee (x > 0).$$

The other three premises establish the conclusion G for all these cases.