Lecture Notes:
Discrete Mathematics for Computer Science

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Part 5. Fibonacci Numbers and Their Relatives

Definition of Fibonacci Numbers

The sequence of *Fibonacci numbers* $F_0, F_1, F_2, \ldots$ is defined by the equations

$$\begin{aligned} &F_0 = 0, \\ &F_1 = 1, \\ &F_{n+2} = F_n + F_{n+1}. \end{aligned}$$

Here the first two members of the sequence are given explicitly, not one. But to calculate any other Fibonacci number we need to know two previous Fibonacci numbers; one is not enough.

The definition of Fibonacci numbers in case notation looks like this:

$$F_n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ F_{n-2} + F_{n-1}, & \text{if } n \geq 2. \end{cases}$$

A Formula for Fibonacci Numbers

We would like to find an explicit formula for Fibonacci numbers. The following terminology will be useful. A *generalized Fibonacci sequence* is a sequence $X_0, X_1, X_2 \ldots$ of real numbers such that for every n,

$$X_{n+2} = X_n + X_{n+1}.$$

We can define a specific generalized Fibonacci sequence by specifying the values of X_0 and X_1. For instance, the values $X_0 = 0$, $X_1 = 1$ will give us the usual Fibonacci numbers

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8, \ldots;$$

if we start with $X_0 = 5$, $X_1 = 7$ then we will get the sequence

$$5,\ 7,\ 12,\ 19,\ 31,\ 50, \ldots. \tag{1}$$

As the first step toward the formula for Fibonacci numbers, we'll find a real number c such that the sequence of its powers

$$1,\ c,\ c^2,\ c^3,\ldots$$

is a generalized Fibonacci sequence. In other words, we are looking for a number c that satisfies the equations

$$\begin{aligned} c^2 &= 1 + c, \\ c^3 &= c + c^2, \\ c^4 &= c^2 + c^3, \\ \ldots \end{aligned}$$

It is sufficient to satisfy the first of these equations, because all other equations follow from it. That equation has two roots:

$$c_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad c_2 = \frac{1-\sqrt{5}}{2} \approx -.618.$$

Thus we determined that the sequences

$$X_n = \left(\frac{1+\sqrt{5}}{2}\right)^n, \qquad Y_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

are generalized Fibonacci sequences.

For any coefficients a and b, the numbers

$$Z_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{2}$$

form a generalized Fibonacci sequence also. Let's now find the values of a and b for which $Z_0 = 0$, $Z_1 = 1$, so that the numbers Z_n become the usual Fibonacci numbers F_n. For $n = 0$ and $n = 1$, we get two equations:

$$\begin{aligned} 0 &= a + b, \\ 1 &= a\frac{1+\sqrt{5}}{2} + b\frac{1-\sqrt{5}}{2}. \end{aligned}$$

From these equations we find:

$$a = \frac{1}{\sqrt{5}}, \quad b = -\frac{1}{\sqrt{5}}.$$

So we arrived at the following formula for Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Modifications of the Fibonacci Sequence

We would like to find an explicit formula for the sequence (1), which is defined by the equations

$$Z_0 = 5,$$
$$Z_1 = 7,$$
$$Z_{n+2} = Z_n + Z_{n+1}.$$

Since the formula expressing Z_{n+2} in terms of Z_n and Z_{n+1} is the same as for Fibonacci numbers, we will look again for a formula of form (2). The equations of a and b are in this case

$$5 = a + b,$$
$$7 = a\frac{1+\sqrt{5}}{2} + b\frac{1-\sqrt{5}}{2}.$$

From these equations we find:

$$a = \frac{25+9\sqrt{5}}{10}, \quad b = \frac{25-9\sqrt{5}}{10}.$$

Consider now the sequence defined by the equations

$$V_0 = 0,$$
$$V_1 = 1,$$
$$V_{n+2} = 3V_n - 2V_{n+1}.$$

To find a number c such that the sequence of its powers $1,\ c,\ c^2,\ c^3, \ldots$ satisfies the last formula, we need to solve the equation $c^2 = 3 - 2c$. Its roots are

$$c_1 = 1, \quad c_2 = -3.$$

So the formula for V_n will have the form

$$V_n = a + b(-3)^n.$$

From the initial conditions $V_0 = 0$, $V_1 = 1$ we find: $a = \frac{1}{4}$, $b = -\frac{1}{4}$.