Lecture Notes:
Discrete Mathematics for Computer Science

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Part 6. Growth of Functions

Comparing the Rates of Growth

Let $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be two increasing sequences of positive numbers. We can decide which of them grows faster by looking at the limit of the ratio $\frac{A_n}{B_n}$ as n goes to infinity, if this limit exists. We say that

$$\begin{aligned} A \text{ grows faster than } B \text{ if} \qquad & \lim \frac{A_n}{B_n} = \infty, \\ A \text{ and } B \text{ grow at the same rate if} \qquad & 0 < \lim \frac{A_n}{B_n} < \infty, \\ B \text{ grows faster than } A \text{ if} \qquad & \lim \frac{A_n}{B_n} = 0. \end{aligned}$$

Here are examples of sequences that grow at different rates:

$\log_2 \log_2 n$	$\log_2 n$	$\sqrt[3]{n}$	$\sqrt{n}$	n	n^2	n^3	1.1^n	2^n	10^n	$n!$	n^n

The difference between their rates of growth can be illustrated by calculating the values of these expressions for $n = 10^6$:

$$\begin{aligned} \log_2 \log_2 n &\approx 4.3 & n^3 &= 10^{18} \\ \log_2 n &\approx 20 & 1.1^n &\approx 10^{40,000} \\ \sqrt[3]{n} &= 100 & 2^n &\approx 10^{300,000} \\ \sqrt{n} &= 1000 & 10^n &= 10^{1,000,000} \\ n &= 10^6 & n! &\approx 10^{5,600,000} \\ n^2 &= 10^{12} & n^n &= 10^{6,000,000} \end{aligned}$$

The Rates of Growth of Logarithms

The relationship between $\log_a n$ and $\log_b n$ is described by the formula

$$\frac{\log_a n}{\log_b n} = \log_a b.$$

In particular,

$$\frac{\ln n}{\log_2 n} = \ln 2 \approx .7$$

and

$$\frac{\log_{10} n}{\log_2 n} = \log_{10} 2 \approx .3.$$

We see that $\log_2 n$, $\ln n$, and $\log_{10} n$ grow at the same rate.

The Rates of Growth of Triangular Numbers and Similar Sequences

We know that

$$T_n = \sum_{i=1}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n,$$

$$S_n = \sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n,$$

and

$$C_n = \sum_{i=1}^{n} i^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2,$$

It follows that

$$\begin{array}{l} T_n \;\text{ grows at the same rate as }\; n^2, \\ S_n \;\text{ grows at the same rate as }\; n^3, \\ C_n \;\text{ grows at the same rate as }\; n^4. \end{array}$$

These are special cases of the general fact: for every real number k that is greater than -1,

$$\sum_{i=1}^{n} i^k \;\text{ grows at the same rate as }\; n^{k+1}.$$

Asymptotically Equal Sequences

About two sequences $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ of positive numbers we say that they are *asymptotically equal* to each other if

$$\lim \frac{A_n}{B_n} = 1.$$

For instance,

$$T_n \text{ is asymptotically equal to } \tfrac{1}{2}n^2,$$

$$S_n \text{ is asymptotically equal to } \tfrac{1}{3}n^3,$$

$$C_n \text{ is asymptotically equal to } \tfrac{1}{4}n^4.$$

These are special cases of the general fact: for every real number k that is greater than -1,

$$\sum_{i=1}^{n} i^k \text{ is asymptotically equal to } \frac{1}{k+1}n^{k+1}.$$

If sequences $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are asymptotically equal, and B_n is easier to calculate than A_n, then B_n can be used as an approximation to A_n when n is large. For example, $\frac{1}{4}n^4$ becomes a good approximation to C_n when n grows, as far as the relative error is concerned:

n	10	20	30	40
Exact value of C_n	3025	44,100	$216,225$	672,400
Approximation $\frac{1}{4}n^4$	2500	40,000	202,500	640,000
Relative error	17%	9%	6%	5%

The Rate of Growth of Fibonacci Numbers

The sequence of Fibonacci numbers F_n grows at the same rate as

$$\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

It is asymptotically equal to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

The Rate of Growth of Harmonic Numbers; Euler's Constant

Recall that the harmonic numbers H_n are defined by the formula

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For instance,

$$H_0 = 0, \quad H_1 = 1, \quad H_2 = \frac{3}{2}, \quad H_3 = \frac{11}{6}, \quad H_4 = \frac{25}{12}.$$

The sequence H_n is asymptotically equal to $\ln n$. When n grows, the difference $H_n - \ln n$ approaches a finite limit, which is called *Euler's constant* and is denoted by γ. It is approximately equal to .577.

The expression $\ln n + \gamma$ is a good approximation to H_n for large values of n. For instance,

$$H_{10} = \frac{7381}{2520} \approx 2.93;$$
$$\ln 10 + \gamma \approx 2.87.$$

The Rate of Growth of Factorials; Stirling's Approximation

The sequence of factorials $n!$ grows approximately as fast as $\left(\frac{n}{e}\right)^n$.

More precisely, $n!$ is asymptotically equal to

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This expression, known as Stirling's approximation, gives a good approximation to $n!$ when n is large, as far as the relative error is concerned. For instance,

$$10! = 3,628,800;$$
$$\sqrt{2\pi \cdot 10}\left(\frac{10}{e}\right)^{10} \approx 3,600,000.$$

The relative error is less than 1%.

Big-O Notation

About two sequences A_n, B_n of positive numbers we say that A_n is $O(B_n)$, and write

$$A_n = O(B_n), \tag{1}$$

if there exist constants C, N such that

$$A_n \le C \cdot B_n \text{ whenever } n \ge N.$$

For instance,

$$2n^2 + 7 = O(n^2),$$

because

$$2n^2 + 7 \le 9 \cdot n^2 \text{ whenever } n \ge 1.$$

Proof: if $n \ge 1$ then

$$2n^2 + 7 \le 2n^2 + 7n^2 = 9n^2.$$