Lecture Notes: Discrete Mathematics for Computer Science

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Part 7. Sets, Relations and Functions

Sets

A *set* is a collection of objects. We write $x \in A$ if object x is an element of set A, and $x \notin A$ otherwise.

The set whose elements are $x_1, \dots, x_n$ is denoted by $\{x_1, \dots, x_n\}$. The set $\{\,\}$ is called *empty* and denoted also by $\emptyset$. The set of nonnegative integers is denoted by $\mathbf{N}$:

$$\mathbf{N} = \{0, 1, 2, \dots\}.$$

Other examples are the set $\mathbf{Z}$ of all integers and the set $\mathbf{R}$ of real numbers.

When we specify which objects belong to a set, this defines the set completely; there is no such thing as the order of elements in a set or the number of repetitions of an element in a set. For instance,

$$\{2, 3\} = \{3, 2\} = \{2, 2, 3\}.$$

If C is a condition, then by $\{x : C\}$ we denote the set of all objects x satisfying this condition. For instance,

$$\{x : x = 2 \lor x = 3\}$$

is the same set as $\{2, 3\}$.

If A is a set and C is a condition, then by $\{x \in A : C\}$ we denote the set of all elements of A satisfying condition C. For instance, $\{2, 3\}$ can be also written as

$$\{x \in \mathbf{N} : 1 < x < 4\}.$$

If a set A is finite then the number of elements of A is also called the *cardinality* of A and denoted by $|A|$. For instance,

$$|\emptyset| = 0, \ |\{2, 3\}| = 2.$$

We say that a set A is a *subset* of a set B, and write $A \subseteq B$, if every element of A is an element of B. For instance,

$$\emptyset \subseteq \mathbf{N}, \ \{2, 3\} \subseteq \mathbf{N}.$$

Operations on Sets

For any sets A and B, by $A \cup B$ we denote the set

$$\{x \;:\; x \in A \;\vee\; x \in B\},$$

called the *union* of A and B. By $A \cap B$ we denote the set

$$\{x \;:\; x \in A \;\wedge\; x \in B\},$$

called the *intersection* of A and B. For instance,

$$\begin{aligned}\{2,3\} \cup \{3,5\} &= \{2,3,5\},\\ \{2,3\} \cap \{3,5\} &= \{3\}.\end{aligned}$$

By $A \setminus B$ we denote the set

$$\{x \;:\; x \in A \;\wedge\; x \notin B\},$$

called the *difference* of A and B. For instance,

$$\{2,3\} \setminus \{3,5\} = \{2\}.$$

The *Cartesian product* of sets A and B is the set of ordered pairs $\langle x, y\rangle$ such that $x \in A$ and $y \in B$:

$$A \times B = \{\langle x, y\rangle \;:\; x \in A \;\wedge\; y \in B\}.$$

For instance,

$$\begin{aligned}\{1,2\} \times \{2,3,4,5,6\} \quad = \quad & \{\langle 1,2\rangle,\ \langle 1,3\rangle,\ \langle 1,4\rangle,\ \langle 1,5\rangle,\ \langle 1,6\rangle,\\ & \ \langle 2,2\rangle,\ \langle 2,3\rangle,\ \langle 2,4\rangle,\ \langle 2,5\rangle,\ \langle 2,6\rangle\}.\end{aligned}$$

By $\mathcal{P}(A)$ we denote the *power set* of a set A, that is, the set of all subsets of A:

$$\mathcal{P}(A) = \{B \;:\; B \subseteq A\}.$$

For instance,

$$\mathcal{P}(\{2,3\}) = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}.$$

Binary Relations

Any condition on a pair of elements of a set A defines a *binary relation*, or simply *relation*, on A. For instance, the condition $x < y$ defines a relation on the set $\mathbf{N}$ of nonnegative integers (or on any other set of numbers). If R is a relation, the formula xRy expresses that R holds for the pair x, y.

A relation R can be characterized by the set of all ordered pairs $\langle x, y\rangle$ such that xRy. It is customary to talk about a relation as it were the same thing as the corresponding set of ordered pairs. For instance, we can say that the relation $<$ on the set $\{1,2,3,4\}$ is the set

$$\{\langle 1,2\rangle,\ \langle 1,3\rangle,\ \langle 1,4\rangle,\ \langle 2,3\rangle,\ \langle 2,4\rangle,\ \langle 3,4\rangle\}.$$

A relation R on a set A is said to be *reflexive* if, for all elements x of A, xRx. We say that R is *irreflexive* if there is no element x of A such that xRx. For instance, the relations $=$ and $\leq$ on the set $\mathbf{N}$ (or on any set of numbers) are reflexive, and the relations $\neq$ and $<$ are irreflexive.

A relation R on a set A is said to be *symmetric* if, for all $x, y \in A$, xRy implies yRx. For instance, the relations $=$ and $\neq$ on $\mathbf{N}$ are symmetric, and the relations $<$ and $\leq$ are not.

A relation R on a set A is said to be *transitive* if, for all $x, y, z \in A$, xRy and yRz imply xRz. For instance, the relations $=$, $<$ and $\leq$ on the set $\mathbf{N}$ are transitive, and the relation $\neq$ is not.

Equivalence Relations and Partitions

An equivalence relation is a relation that is reflexive, symmetric, and transitive.

A *partition* of a set A is a collection P of non-empty subsets of A such that every element of A belongs to exactly one of these subsets. For instance, here are some partitions of $\mathbf{N}$:

$$\begin{aligned} P_1 &= \{\{0, 2, 4, \dots\}, \{1, 3, 5, \dots\}\}, \\ P_2 &= \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \dots\}, \\ P_3 &= \{\{0\}, \{1\}, \{2\}, \{3\}, \dots\}. \end{aligned}$$

If P is a partition of a set A then the relation "x and y belong to the same element of P" is an equivalence relation.

Order Relations

A relation R on a set A is said to be *antisymmetric* if, for all $x, y \in A$, xRy and yRx imply $x = y$. For instance, the relation $\leq$ on $\mathbf{R}$ is antisymmetric.

A *partial order* is a relation that is reflexive, anti-symmetric, and transitive. For instance, the relation $\leq$ on $\mathbf{R}$, the relation $|$ on $\mathbf{N}$, and the relation $\subseteq$ on $\mathcal{P}(A)$ for any set A are partial orders.

A *total order* on a set A is a partial order such that for all $x, y \in A$, xRy or yRx. For instance, $\leq$ is total and $|$ is not.

General Definition of a Function

For any sets A and B, a *function from A to B* is a rule f that can be applied to any element x of A and produces an element $f(x)$ of B. The set A is called the *domain* of f. The subset of B consisiting of the values $f(x)$ of the function for all $x \in A$ is called the *range* of f. If the range of f is the whole set B then we say that f is a function *onto* B.

This definition of a function is general because it does not assume that the domain and the range consist of numbers. In the following examples, the domain of each function is the set $\mathbf{S}$ of all bit strings:

$$\mathbf{S} = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \dots\}.$$

1. Function l from $\mathbf{S}$ to $\mathbf{N}$: $l(x)$ is the length of x. For instance, $l(00110) = 5$.

2. Function z from $\mathbf{S}$ to $\mathbf{N}$: $z(x)$ is the number zeroes in x. For instance, $z(00110) = 3$.

3. Function n from $\mathbf{S}$ to $\mathbf{N}$: $n(x)$ is the number represented by x in binary notation. For instance, $n(00110) = 6$.

4. Function e from $\mathbf{S}$ to $\mathbf{S}$: $e(x)$ is the string $1x$. For instance, $e(00110) = 100110$.

5. Function r from $\mathbf{S}$ to $\mathbf{S}$: $r(x)$ is the string x reversed. For instance, $r(00110) = 01100$.

6. Function p from $\mathbf{S}$ to $\mathcal{P}(\mathbf{S})$: $p(x)$ is the set of prefixes of x. For instance, $p(00110) = \{\epsilon, 0, 00, 001, 0011, 00110\}$.

A function f can be characterized by the set of all ordered pairs of the form $\langle x, f(x)\rangle$. It is customary to talk about a function as if it were the same thing as the corresponding set of ordered pairs. For instance, we can say that the function f from $\mathbf{N}$ to $\mathbf{N}$ defined by the formula $f(n) = 2n + 1$ is the set

$$\{\langle 0,1\rangle,\ \langle 1,3\rangle,\ \langle 2,5\rangle, \dots\}.$$

Instead of defining functions as rules, we can say that a function from a set A to a set B is a set $f \subseteq A \times B$ such that for every element x of A there exists a unique element y of B for which $\langle x, y\rangle \in f$.

A function f from A to B is called *one-to-one* if, for any pair of different elements x, y of A, $f(x)$ is different from $f(y)$. If a function f is both onto and one-to-one then we say that f is a *bijection*. A *permutation* of a set A is a bijection from A to A.

If f is a function from A to B, and g is a function from B to C, then the *composition* of these functions is the function h from A to C defined by the formula $h(x) = g(f(x))$. This function is denoted by $g \circ f$.

If f is a bijection from A to B then the *inverse* of f is the function g from B to A such that, for every $x \in A$, $g(f(x)) = x$. This function is denoted by f^{-1}.