Lecture Notes: Discrete Mathematics for Computer Science

Vladimir Lifschitz
University of Texas at Austin

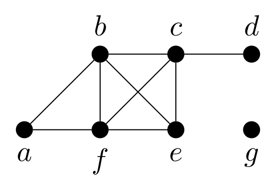
Part 8. Graphs

Undirected Graphs

An *undirected graph* G is defined by two sets: a set V of objects called the *vertices* (or *nodes*) of G, and a collection E of two-element subsets $\{u, v\}$ of V, called the *edges* of the graph. The vertices u, v are called the *ends* of the edge $\{u, v\}$. When we draw a graph, we usually show every vertex as a small circle, and every edge as a line segment joining its two ends. For instance, the graph with the vertices a, b, c, d, e, f, g and the edges

$$\{a, b\},\ \{b, c\},\ \{c, d\},\ \{a, f\},\ \{f, e\},\ \{b, e\},\ \{c, f\},\ \{b, f\},\ \{c, e\}$$

looks like this:



If a graph has an edge with the ends u, v, we say that u and v are *adjacent*. The adjacency relation is symmetric and irreflexive. The number of vertices adjacent to a vertex v is called the *degree* of v. For instance, the degree of a in the graph shown above is 2, and the degree of g is 0. Vertices of degree 0, such as g, are called *isolated*.

The *adjacency matrix* of a graph with n vertices is the $n \times n$ matrix such that its entry in row u and column v is 1 if u, v are adjacent, and 0 otherwise. For instance, here is the adjacency matrix of the graph shown above:

	a	b	c	d	e	f	g
a	0	1	0	0	0	1	0
b	1	0	1	0	1	1	0
c	0	1	0	1	1	0	0
d	0	0	1	0	0	0	0
e	0	1	1	0	0	1	0
f	1	1	0	0	1	0	0
g	0	0	0	0	0	0	0

The adjacency matrix of any graph is symmetric, and its main diagonal consists of zeroes.

A *complete graph on n vertices*, denoted by K_n, contains an edge between each pair of vertices.

A graph is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 so that every edge has one vertex in V_1 and one in V_2.

Paths in a Graph

A *path* in a graph is a list

$$v_1, v_2, \ldots, v_k \tag{1}$$

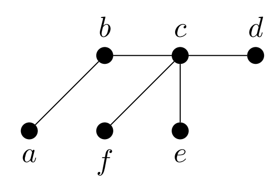
of vertices such that every two consecutive vertices v_i, v_{i+1} are adjacent. About this path we say that it is a path *from v_1 to v_k*. A path is *simple* if its vertices are distinct from one another. A path (1) is a *cycle* if $k > 2$, $v_k = v_1$, the vertices $v_1, v_2, \ldots, v_{k-1}$ are all distinct, and the edges $\{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}$ are all distinct. For instance, a, b, c, e is a simple path in the graph above, and a, b, c, e, f, a is a cycle in that graph.

The relation "there is a path from u to v" is an equivalence relation. Its equivalence classes are called the *connected components* of the graph. For instance, the connected components of the graph above are $\{a, b, c, d, e, f\}$ and $\{g\}$. If u and v belong to the same connected component then the distance from u to v is defined as the minimum number of edges in a path from u to v. For instance, the distance from a to c is 2.

If for every pair of vertices u, v there is a path from u to v then we say that the graph is *connected*.

Trees

A *tree* is a connected graph that doesn't contain cycles. For instance, the graph



is a tree.

A *rooted tree* is a tree with one distinguished vertex, called the *root*. For every vertex v in a rooted tree, there is unique path from the root to v, The vertices that belong to that path are called the *ancestors* of x; if the last edge of that path is $\{y, x\}$ then we say that y is the *parent* of x, and x is a *child* of y. A vertex that doesn't have children is called a *leaf*.

A tree with n vertices has $n - 1$ edges: for every vertex v other than the root, the corresponding edge connects v with its parent.

Directed Graphs

In a *directed graph*, edges are ordered pairs of vertices, rather than two-element subsets. About an edge $\langle u, v \rangle$ we say that it *leaves* u and *enters* v. When we draw a directed graph, we show an edge $\langle u, v \rangle$ as an arrow from u to v.

The number of edges that enter a vertex v is called the *in-degree* of v. The number of edges that leave v is called the *out-degree* of v. The *adjacency matrix* of a directed graph with n vertices is the $n \times n$ matrix such that its entry in row u and column v is 1 if there graph has an edge that leaves u and enters v, and 0 otherwise.

A *path* in a directed graph is a list (1) of vertices such that for every two consecutive vertices v_i, v_{i+1} the graph has an edge that leaves v_i and enters v_{i+1}.

The relation "there is a path from u to v and a path from v to u" is an equivalence relation. Its equivalence classes are called the *strongly connected components* of the graph.