CS311: Discrete Math for Computer Science, Spring 2015

Test 3, with Solutions

Open notes. No books, no calculators. Justify your answers.

1. Determine which of the two numbers is greater:

(a) $90!$ or 30^{80}.

Solution: $90! \approx \left(\frac{90}{e}\right)^{90} > 30^{90} > 30^{80}$.

(b) H_{10000} or $2 \cdot H_{100}$.

Solution:

$$\begin{aligned} H_{10000} &\approx \ln 10000 + \gamma = 2\ln 100 + \gamma; \\ 2H_{100} &\approx 2(\ln 100 + \gamma) = 2\ln 100 + 2\gamma. \end{aligned}$$

We conclude that $2H_{100}$ is greater.

2. Prove that

(a) $(n+1)^3 = O(n^3)$,

Solution: Take $C = 8, N = 1$. Assuming that $n \geq N$,

$$(n+1)^3 \leq (2n)^3 \leq 8n^3.$$

(b) $A_n = O(n)$, where

$$A_n = \begin{cases} 100, & \text{if } n < 10, \\ n+1, & \text{otherwise.} \end{cases}$$

Solution: Take $C = 2, N = 10$. Assuming that $n \geq N$,

$$A_n = n + 1 < n + n = 2n.$$

3. Find the cardinalities of the following sets:

(a) $\{x \in \mathbf{R} : x^2 < 5\} \cap \mathbf{Z}$,

Solution: $|\{x \in \mathbf{R} : x^2 < 5\} \cap \mathbf{Z}| = |\{-2, -1, 0, 1, 2\}| = 5$.

(b) $(\{1, \ldots, 10\} \times \{1, \ldots, 10\}) \cup (\{10, \ldots, 20\} \times \{10, \ldots, 20\})$,

Solution. The set $\{1, \ldots, 10\} \times \{1, \ldots, 10\}$ has 100 elements; the set $\{10, \ldots, 20\} \times \{10, \ldots, 20\}$ has 121 elements. These sets have one common element—the pair $\langle 10, 10 \rangle$. It follows that the cardinality of the union is $100 + 121 - 1$, that is, 220.

(c) $\mathcal{P}(\{1,2,3\}) \setminus \mathcal{P}(\{1,3\})$.

Solution. This set consists of the subsets of $\{1,2,3\}$ that are not among the subsets of {1,3}. There are 4 such sets:

$$\{2\},\ \{1,2\},\ \{2,3\},\ \{1,2,3\}.$$

4. Determine whether the relation $m = -n$ on the set of integers is (a) reflexive, (b) irreflexive, (c) symmetric, (d) transitive.

Solution. This relation R is not reflexive, because $1R1$ is not true. It is not irreflexive, because $0R0$ is true. It is symmetric, because the condition $m = -n$ is equivalent to $n = -m$. It is not transitive, because the conditions $1R{-1}$ and $-1R1$ hold, but the condition $1R1$ doesn't.