Abstract Gringo

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Abstract
This paper defines the syntax and semantics of the input language of the ASP grounder GRINGO. The definition covers several constructs that were not discussed in earlier work on the semantics of that language, including intervals, pools, division of integers, aggregates with non-numeric values, and lparse-style aggregate expressions. The definition is abstract in the sense that it disregards some details related to representing programs by strings of ASCII characters. It serves as a specification for GRINGO from Version 4.5 on.

1 Introduction

Version 4.0 of the ASP grounder GRINGO was released in March of 2013.\footnote{http://potassco.sourceforge.net/} Harrison et al. [2014] defined the semantics of a subset of its input language in terms of stable models of infinitary propositional formulas [Truszczynski, 2012]. That subset does not include, however, several constructs that are frequently used in ASP programs. One such construct is integer intervals. Take, for instance, the ASP solution to the n-queens problem shown in Table 1. (It is similar to one of the solutions in the language of Version 3 presented by Gebser et al. [2011].) Intervals are used in each rule of this program.

To include intervals, we have to modify the semantics from Harrison et al. [2014] in two ways. First, we have to say that an arithmetic term denotes, generally, a finite set of integers, not a single integer. (And it is not necessarily a set of consecutive integers, because the language of GRINGO allows us to write \((1..3)*2\), for instance. This expression denotes the set \(\{2, 4, 6\}\).) Second, in the presence of intervals we cannot treat a choice rule \(\{A\}\) as shorthand for the disjunctive rule \(A ; \text{not } A\) as proposed by Ferraris and Lifschitz [2005]. Indeed, the first rule of the program in Table 1 has \(2^n\)\(^2\) stable models; the rule
\[ q(1..n,1..n) ; \text{not } q(1..n,1..n) \]
has only 2 stable models.

Another feature of GRINGO not covered by Harrison et al. [2014], which is somewhat similar to integer intervals, is pooling. Pooling is used, for instance, in the head of the rule
\[ p(X;Y) :- q(X,Y). \]
% place queens on the chess board
\{ q(1..n,1..n) \}.

% exactly 1 queen per row/column
:- X = 1..n, not #count\{ Y : q(X,Y) \} = 1.
:- Y = 1..n, not #count\{ X : q(X,Y) \} = 1.

% pre-calculate the diagonals
d1(X,Y,X-Y+n) :- X = 1..n, Y = 1..n.
d2(X,Y,X+Y-1) :- X = 1..n, Y = 1..n.

% at most one queen per diagonal
:- D = 1..n*2-1, 2 \{ q(X,Y) : d1(X,Y,D) \}.
:- D = 1..n*2-1, 2 \{ q(X,Y) : d2(X,Y,D) \}.

---

Table 1: An ASP solution to the \( n \)-queens problem.

(Note that a semicolon, not a comma, separates \( X \) from \( Y \) in the head.) This rule has the same meaning as the pair of rules

\[
\begin{align*}
p(X) & :- q(X,Y). \\
p(Y) & :- q(X,Y).
\end{align*}
\]

Pooling is often used to abbreviate a set of facts. For instance, instead of

\[
\begin{align*}
p(a,5). & \quad p(b,10). \quad p(c,12). \\
p(a,5; b,10; c,12).
\end{align*}
\]

we can write \( p(a,5; b,10; c,12) \). In this paper, we talk about “pools”—groups of terms such as \( a,5; b,10; c,12 \).

Yet another limitation of the proposal from Harrison et al. [2014] is related to the difference between “dlv-style” aggregates, such as

\[
\text{not \ #count}\{ Y : q(X,Y) \} = 1
\]  \hspace{1cm} (1)

in the second rule of the program shown in Table 1, and “lpars-style” aggregates, such as

\[
2 \{ q(X,Y) : d2(X,Y,D) \}
\]  \hspace{1cm} (2)

in the last rule of the program. Both expressions have to do with counting. Syntactically, the difference is that in expression (1) both the name of the aggregate (**#count**) and the binary relation applied to the result of counting and a constant (=) are shown explicitly; in (2), the fact that the constant 2 occurs on the left, in the lower bound position, tells us that the relation \( \leq \) is applied to that number and to the result of counting. More importantly, there is a difference
between the kinds of objects that we count. In case of expression (1) we count, for a given value
of \( X \), the values of the variable \( Y \) such that \( q(X,Y) \) belongs to the stable model. In case of (2)
we count, for a given value of \( D \), the atoms \( q(X,Y) \) that belong to the stable model and satisfy
an additional condition: \( d_2(X,Y,D) \) belongs to the model as well. Thus the atom in front of
the colon in (2) plays two roles: it tells us what to count, and it gives a condition on the stable
model that needs to be checked.

The language studied by Harrison et al. [2014] does not include lparse-style aggregates.
The easiest way to add such aggregates is to treat them as abbreviations. For instance, (2) can
be viewed as shorthand for the dlv-style expression

\[
2 \leq \text{#count}\{ q(X,Y) : q(X,Y), d_2(X,Y,D) \}.
\]

(In this paper we adopt a more elaborate translation that allows us to accommodate negated
atoms in front of the colon.) In this expression, the first occurrence of \( q(X,Y) \) is syntactically
a term, and the second is an atom. Thus treating (2) as an abbreviation depends on the possibility
of using the same symbol as a function and as a predicate. This is customary in Prolog, but not
in first-order logic, and this was not allowed by Harrison et al. [2014].

Our goal is to define the syntax and semantics of the language AG (short for Abstract
Gringo)—a large subset of the input language of GRINGO that includes the features mentioned
previously and a few other constructs not described by Harrison et al. [2014]. This is similar to
the work that has led to the definition of the ASP Core language [Calimeri et al., 2012].\(^2\),\(^3\)

The semantics of AG defined in this paper serves a specification for GRINGO from Ver-
sion 4.5 on. It can be used to prove the correctness of programs written in its input language.
As an example, in the electronic appendix we prove the correctness of the program shown in
Table 1.

AG is abstract in the sense that its definition disregards some details related to representing
programs by strings of ASCII characters. For example, semicolons are used in the input lan-
guage of GRINGO in at least three ways: to denote disjunction in the head of a rule, conjunction
in the body, and pooling within an atom. In AG, three different symbols play these different
roles. The richer alphabet of AG makes it easier to define the semantics of the language and to
reason about ASP programs.

\(^2\)Syntactically, AG is essentially an extension of ASP Core. But it does not include extra-logical constructs, such
as weak constraints and queries. The semantics of aggregates in AG is based on the approach of Ferraris [2005]
and thus is not equivalent to the semantics of aggregates in ASP Core when aggregates are used recursively in the
presence of negation. Among the language constructs that are not in ASP Core, in AG we find pooling, intervals,
and conditional literals. (These constructs originally appeared in the input language of LPARSE, but in AG they
are more general; for instance, interval bounds may contain variables, and restrictions to “domain predicates” have
disappeared.) Unlike ASP Core, AG supports aggregates in rule heads; see Section 3.

\(^3\)The definition of ASP Core does not refer to infinitary objects, such as infinitary propositional formulas used in
this paper. But it appears that infinitary objects of some kind will be required to correct the oversight in [Calimeri et
al., 2012, Section 2.2]—the set \( \text{inst}\{e_1;\ldots;e_n\} \), included in the body of a rule in the process of instantiation, can
be infinite.

\[3\]
2 Syntax of AG

2.1 Symbols and Terms

We assume that five sets of symbols are selected: numerals, symbolic constants, negated constants, variables, and aggregate names. We assume that a 1–1 correspondence between the set of symbolic constants and the set of negated constants is chosen. For every symbolic constant \( p \), the corresponding negated constant will be called its strong negation and denoted by \( \tilde{p} \). Further, we assume that these sets do not contain the symbols

\[
\begin{align*}
+ & - \times / . \\
\inf & \sup \\
= & \neq < > \leq \geq \\
\bot & not \land \lor \leftarrow \\
, & ; : ( ) \{ \} \langle \rangle
\end{align*}
\]

and that they are pairwise disjoint. All these symbols together form the alphabet of AG, and AG rules will be defined as strings over this alphabet.

When a symbol is represented in ASCII, its type is determined by its first two characters. For instance, a numeral starts with a digit or – followed by a digit. A symbolic constant starts with a lower-case letter. A negated constant starts with – followed by a lower-case letter. A variable starts with an upper-case letter, and an aggregate name starts with #. (The strings \#false, \#inf, and \#sup, which represent \( \bot \), \( \inf \), and \( \sup \), also start with #.) The symbols \( ⟨ \) and \( ⟩ \) (which are used to indicate the boundaries of a tuple within a term) correspond to the ASCII characters \( ( \) and \( ) \). Each of the symbols (3)–(7) except for \( \land \) and \( \lor \) has a unique ASCII representation; the symbols \( \land \) and \( \lor \) can be represented by semicolons and in some cases also by commas.

We assume that a 1–1 correspondence between the set of numerals and the set \( \mathbb{Z} \) of integers is chosen. For every integer \( n \), the corresponding numeral will be denoted by \( \overline{n} \).

Terms are defined recursively, as follows:

- all numerals, symbolic constants, and variables are terms;
- if \( f \) is a symbolic constant and \( t \) is a tuple of terms then \( f(t) \) is a term;
- if \( t_1 \) and \( t_2 \) are terms and \( * \) is one of the symbols (3) then \( (t_1 * t_2) \) is a term;
- if \( t \) is a tuple of terms then \( \langle t \rangle \) is a term.

\( ^4 \)When an AG term representing a tuple of length 1, such as \( \langle a \rangle \), is represented in ASCII, a comma is appended to the tuple: \( \langle a, \rangle \).

\( ^5 \)In this paper, when we refer to a tuple of syntactic objects, we mean that the tuple may be empty and that its members are separated by commas.
In a term of the form \( f() \) the parentheses can be dropped, so that every symbolic constant can be viewed as a term. In a term of the form \((t_1 \cdot t_2)\) we will drop the parentheses when it should not lead to confusion. A term of the form \((\overline{0} - t)\) can be abbreviated as \(-t\).

A term, or a tuple of terms, is precomputed if it contains neither variables nor symbols (3). We assume a total order on precomputed terms such that \(\inf\) is its least element, \(\sup\) is its greatest element, and, for any integers \(m\) and \(n\), \(\overline{m} \leq \overline{n}\) iff \(m \leq n\).

We assume that for each aggregate name \(\alpha\) a function \(\hat{\alpha}\) is chosen that maps every set of tuples of precomputed terms to a precomputed term.\(^6\) The AG counterparts of the aggregates implemented in Version 4.5 of GRINGO are defined below using the following terminology. If the first member of a tuple \(t\) of precomputed terms is a numeral \(n\) then we say that the integer \(n\) is the weight of \(t\); if \(t\) is empty or its first member is not an integer then the weight of \(t\) is 0. For any set \(T\) of tuples of precomputed terms,

- \(\hat{\text{count}}(T)\) is the numeral corresponding to the cardinality of \(T\) if \(T\) is finite, and \(\sup\) otherwise;
- \(\hat{\text{sum}}(T)\) is the numeral corresponding to the sum of the weights of all tuples in \(T\) if \(T\) contains finitely many tuples with non-zero weights, and 0 otherwise;
- \(\hat{\text{sum}+}(T)\) is the numeral corresponding to the sum of the weights of all tuples in \(T\) whose weights are positive if \(T\) contains finitely many such tuples, and \(\sup\) otherwise;
- \(\hat{\text{min}}(T)\) is \(\sup\) if \(T\) is empty, the least element of the set consisting of the first elements of the tuples in \(T\) if \(T\) is a finite non-empty set, and \(\inf\) if \(T\) is infinite;
- \(\hat{\text{max}}(T)\) is \(\inf\) if \(T\) is empty, the greatest element of the set consisting of the first elements of the tuples in \(T\) if \(T\) is a finite non-empty set, and \(\sup\) if \(T\) is infinite.

2.2 Atoms, Literals, and Choice Expressions

A pool is an expression of the form \(t_1; \ldots; t_n\) where \(n \geq 1\) and each \(t_i\) is a tuple of terms.\(^7\) In particular, every tuple of terms is a pool.

An atom is a string of one of the forms \(p(P)\), \(\overline{\tilde{p}}(P)\) where \(p\) is a symbolic constant and \(P\) is a pool. In an atom of the form \(p()\) or \(\overline{\tilde{p}}()\) the parentheses can be dropped, so that all symbolic constants and all negated constants can be viewed as atoms.

For any atom \(A\), the strings

\[
A \quad \text{not } A \quad \text{not not } A
\]

\(^6\)This understanding of \(\hat{\alpha}\) is different from that given by Harrison et al. [2014]. There, \(\hat{\alpha}\) is understood as a function that maps tuples of precomputed terms to elements of \(\mathbb{Z} \cup \{\infty, -\infty\}\).

\(^7\)This form of pooling is less general than what is allowed in the input language of GRINGO. For instance, \(f(a; b)\) is neither a term nor a pool.
are symbolic literals. An arithmetic literal is a string of the form \( t_1 \prec t_2 \) where \( t_1, t_2 \) are terms and \( \prec \) is one of the symbols (5)

A conditional literal is a string of the form \( H : L \) where \( H \) is a symbolic or arithmetic literal or the symbol \( \bot \) and \( L \) is a tuple of symbolic or arithmetic literals. If \( L \) is empty then we will drop the colon, so that every symbolic or arithmetic literal can be viewed as a conditional literal.

An aggregate atom is a string of one of the forms

\[
\begin{align*}
\alpha \{ t_1 : L_1 ; \ldots ; t_n : L_n \} \prec s \\
s \prec \alpha \{ t_1 : L_1 ; \ldots ; t_n : L_n \} \\
s_1 \prec s \prec s_2
\end{align*}
\]

\((n \geq 0)\), where

- \( \alpha \) is an aggregate name,
- each \( t_i \) is a tuple of terms,
- each \( L_i \) is a tuple of symbolic or arithmetic literals (if \( L_i \) is empty and \( t_i \) is nonempty then the preceding colon may be dropped),
- each of \( \prec, \prec_1, \prec_2 \) is one of the symbols (5),
- each of \( s, s_1, s_2 \) is a term.

For any aggregate atom \( A \), the strings (8) are aggregate literals.

A literal is a conditional literal or an aggregate literal.

A choice expression is a string of the form \( \{ A \} \) where \( A \) is an atom.

### 2.3 Rules and Programs

A rule is a string of the form

\[
H_1 \lor \cdots \lor H_k \leftarrow B_1 \land \cdots \land B_m
\]

or of the form

\[
C \leftarrow B_1 \land \cdots \land B_m
\]

\((k, m \geq 0)\), where each \( H_i \) is a symbolic or arithmetic literal,\(^9\) \( C \) is a choice expression, and each \( B_j \) is a literal. The expression \( B_1 \land \cdots \land B_m \) is the body of the rule; \( H_1 \lor \cdots \lor H_k \) is the

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\(^8\)Semantically, the status of “double negations” in AG is the same as in logic programs with nested expressions [Lifschitz et al., 1999], where conjunction, disjunction, and negation can be nested arbitrarily. Dropping a double negation may change the meaning of a rule. For instance, the one-rule program \( p \leftarrow \neg \neg p \) has two stable models \( \emptyset, \{ p \} \) (see Section 4.1); the latter will disappear if we drop \( \neg \neg \).

\(^9\)In the input language of GRINGO, dropping the colon when \( L \) is empty is required.

\(^{10}\)In the input language of GRINGO, \( H_i \) may be any conditional literal.
head of (12); C is the head of (13). If the body of a rule is empty and the head is not then the arrow can be dropped.

For instance, here are the first five rules of the program from Table 1 written in the syntax of AG:

\[
R_1 : \{ \text{\(q(T \ldots \pi, T \ldots \pi)\)} \}, \\
R_2 : \leftarrow \text{\(X = T \ldots \pi \wedge \text{not count}\{Y : q(X, Y)\} = T\)}, \\
R_3 : \leftarrow \text{\(Y = T \ldots \pi \wedge \text{not count}\{X : q(X, Y)\} = T\)}, \\
R_4 : \text{\(d1(X, Y, X - Y + \pi) \leftarrow X = T \ldots \pi \wedge Y = T \ldots \pi\)}, \\
R_5 : \text{\(d2(X, Y, X + Y - T) \leftarrow X = T \ldots \pi \wedge Y = T \ldots \pi\)}.
\]

The other two rules use abbreviations introduced in the next section.

A program is a finite set of rules.

3 Abbreviations

Let C be an expression of the form

\[
s_1 \prec_1 \alpha\{t_1 : L_1 ; \ldots ; t_n : L_n\} \prec_2 s_2
\]

\((n \geq 0)\), where each \(L_i\) is a symbolic literal of one of the forms

\[
p(t), \quad \text{not} \ p(t), \quad \text{not not} \ p(t)
\]

\((p\) is a symbolic or negated constant and \(t\) is a tuple of terms\) and \(\alpha, t_i, L_i, \prec_1, \prec_2, s_1,\) and \(s_2\) are as in the definition of an aggregate atom. Then a string of the form

\[
C \leftarrow B_1 \wedge \cdots \wedge B_m
\]

\((m \geq 0)\), where each \(B_j\) is a literal, is shorthand for the set of rules consisting of the rule

\[
\leftarrow B_1 \wedge \cdots \wedge B_m \wedge \text{not} \ s_1 \prec_1 \alpha\{t_1 : L_1 ; \ldots ; t_n : L_n\} \prec_2 s_2
\]

and, for each \(L_i\) in (14) such that \(L_i\) is an atom, the rule

\[
\{L_i\} \leftarrow B_1 \wedge \cdots \wedge B_m \wedge C_i
\]

where \(C_i\) is the conjunction of the members of \(L_i\).

In both (17) and (18), the conjunction sign shown after \(B_m\) should be dropped if \(m = 0\); in (18) it should also be dropped if \(C_i\) is empty. The parts \(s_1 \prec_1\) and \(\prec_2 s_2\) in (14) are optional; if one of them is missing then it is dropped from (17) as well; if both are missing then rule (17) is dropped from the set altogether. If \(m = 0\) in (16) then the arrow can be dropped.

The term representations of literals (15) are the tuples

\[
\overline{0}, p(t), \quad \overline{1}, p(t), \quad \overline{2}, p(t)
\]
of terms. (Each of them is indeed a tuple of terms, because \( p(t) \) can be viewed as a term.)

Also viewed as an abbreviation is any expression of the form

\[
s_1 \{ L_1 : L_1 ; \ldots ; L_n : L_n \} s_2
\]

\((n > 0)\), where \( s_1, s_2 \) are terms, each \( L_i \) is a symbolic literal of one of the forms (15) that does not contain \( \ldots \), and each \( L_i \) is a tuple of symbolic or arithmetic literals.\(^{11}\) Such an expression is understood differently depending on whether it occurs in the head or the body of a rule. In the head of a rule, (19) is understood as shorthand for an expression of the form (14):

\[
s_1 \leq \text{count}\{ t_1 : L_1 ; \ldots ; t_n : L_n \} \leq s_2
\]

where \( t_i \) is the term representation of \( L_i \). If either or both of the terms \( s_1, s_2 \) are missing, the abbreviation is understood in a similar way. (Note that choice expressions that do not contain \( \ldots \) are expressions of the form (19) where both \( s_1 \) and \( s_2 \) are missing, \( n = 1 \), \( L_1 \) is of the form \( p(t) \), and \( L_1 \) is empty. In this case, we do not view (19) as an abbreviation.)

In the body of a rule (19) is understood as shorthand for the aggregate atom

\[
s_1 \leq \text{count}\{ t_1 : L_1, L_1; \ldots ; t_n : L_n, L_n \} \leq s_2
\]

where \( t_i \) is the term representation of \( L_i \).\(^{12}\) If either of the terms \( s_1, s_2 \) in (19) is missing, the abbreviation is understood in a similar way.

These abbreviations can be used, for instance, to represent the last two rules of the program from Table 1 in the syntax of AG:

\[
\begin{align*}
\leftarrow D &= \top \ldots \top \times \overline{2} - \top \land \overline{2} \{ q(X, Y) : d1(X, Y, D) \}, \\
\leftarrow D &= \top \ldots \top \times \overline{2} - \top \land \overline{2} \{ q(X, Y) : d2(X, Y, D) \}.
\end{align*}
\]

Written out in full, these expressions become

\[
\begin{align*}
R_6 & : \quad \leftarrow D = \top \ldots \top \times \overline{2} - \top \land \overline{2} \leq \text{count}\{ \overline{0}, q(X, Y) : q(X, Y), d1(X, Y, D) \}, \\
R_7 & : \quad \leftarrow D = \top \ldots \top \times \overline{2} - \top \land \overline{2} \leq \text{count}\{ \overline{0}, q(X, Y) : q(X, Y), d2(X, Y, D) \}.
\end{align*}
\]

4 Semantics of AG

We will define the semantics of AG using a syntactic transformation \( \tau \). The function \( \tau \) converts rules into infinitary formulas formed from atoms of the form \( p(t) \) or \( \overline{p}(t) \), where \( p \) is a symbolic constant, and \( t \) is a tuple of precomputed terms. Then the stable models of a program will be defined in terms of stable model semantics of infinitary formulas in the sense of Truszczynski [2012], which is reviewed below.

\(^{11}\)To be precise, if \( L_1 \) is empty then the colon after \( L_1 \) is dropped.

\(^{12}\)If \( L_n \) is empty then the comma after \( L_n \) in this expression should be dropped.
4.1 Review: Infinitary Propositional Formulas

Let $\sigma$ be a propositional signature, that is, a set of propositional atoms. The sets $\mathcal{F}_0, \mathcal{F}_1, \ldots$ are defined as follows:

- $\mathcal{F}_0 = \sigma$,
- $\mathcal{F}_{i+1}$ is obtained from $\mathcal{F}_i$ by adding expressions $\mathcal{H}^\land$ and $\mathcal{H}^\lor$ for all subsets $\mathcal{H}$ of $\mathcal{F}_i$, and expressions $F \rightarrow G$ for all $F, G \in \mathcal{F}_i$.

The elements of $\bigcup_{i=0}^{\infty} \mathcal{F}_i$ are called (infinitary) formulas over $\sigma$.

In an infinitary formula, the symbols $\top$ and $\bot$ are understood as abbreviations for $\emptyset^\land$ and $\emptyset^\lor$ respectively; $\neg F$ stands for $F \rightarrow \bot$, and $F \leftrightarrow G$ stands for $(F \rightarrow G) \land (G \rightarrow F)$.

Subsets of a signature $\sigma$ will also be called its interpretations. The satisfaction relation between an interpretation and a formula is defined recursively as follows:

- For every atom $p$ from $\sigma$, $I | = p$ if $p \in I$.
- $I | = \mathcal{H}^\land$ if for every formula $F$ in $\mathcal{H}$, $I | = F$.
- $I | = \mathcal{H}^\lor$ if there is a formula $F$ in $\mathcal{H}$ such that $I | = F$.
- $I | = F \rightarrow G$ if $I \neq F$ or $I | = G$.

We say that an interpretation satisfies a set $\mathcal{H}$ of formulas, or is a model of $\mathcal{H}$, if it satisfies every formula in $\mathcal{H}$. Two sets of formulas are equivalent if they have the same models.

The reduct $F^I$ of a formula $F$ w.r.t. an interpretation $I$ is defined as follows:

- For $p \in \sigma$, $p^I = \bot$ if $I \neq p$; otherwise $p^I = p$.
- $(\mathcal{H}^\land)^I = \{ G^I \mid G \in \mathcal{H} \}^\land$.
- $(\mathcal{H}^\lor)^I = \{ G^I \mid G \in \mathcal{H} \}^\lor$.
- $(G \rightarrow H)^I = \bot$ if $I \neq G \rightarrow H$; otherwise $(G \rightarrow H)^I = G^I \rightarrow H^I$.

An interpretation $I$ is a stable model of a set $\mathcal{H}$ of formulas if it is minimal w.r.t. set inclusion among the interpretations satisfying the reducts of all formulas from $\mathcal{H}$.

For instance, if $I = \emptyset$ then

$$(-p \rightarrow p)^I = (-p)^I \rightarrow p^I = \bot \rightarrow \bot;$$

if $I = \{p\}$ then

$$(-p \rightarrow p)^I = (-p)^I \rightarrow p^I = -(-p)^I \rightarrow p = -\bot \rightarrow p.$$

In both cases, $I$ is a minimal model of the reduct. Consequently, both $\emptyset$ and $\{p\}$ are stable models of $\{-p \rightarrow p\}$.
4.2 Semantics of Terms and Pools

A term is ground if it does not contain variables. The definition of “ground” for pools, symbolic literals, and arithmetic literals is the same. Semantically, every ground term \( t \) represents a finite set of precomputed terms \([t]\), which is defined recursively:

- if \( t \) is a numeral or a symbolic constant then \([t]\) is \( \{t\} \);
- if \( t = f(t_1, \ldots, t_n) \) then \([t]\) is the set of terms \( f(r_1, \ldots, r_n) \) for all \( r_1 \in [t_1], \ldots, r_n \in [t_n] \);
- if \( t = (t_1 + t_2) \) then \([t]\) is the set of numerals \( n_1 + n_2 \) for all integers \( n_1, n_2 \) such that \( \overline{m_1} \in [t_1] \) and \( \overline{m_2} \in [t_2] \); similarly when \( t = (t_1 - t_2) \) or \( (t_1 \times t_2) \);
- if \( t = (t_1/t_2) \) then \([t]\) is the set of numerals \( n_1/n_2 \) for all integers \( n_1, n_2 \) such that \( \overline{m_1} \in [t_1], \overline{m_2} \in [t_2], \) and \( n_2 \neq 0 \);
- if \( t = (t_1 \ldots t_2) \) then \([t]\) is the set of numerals \( m \) for all integers \( m \) such that, for some integers \( n_1, n_2 \),
  \[
  \overline{m_1} \in [t_1], \quad \overline{m_2} \in [t_2], \quad n_1 \leq m \leq n_2;
  \]
- if \( t = \langle t_1, \ldots, t_n \rangle \) then \([t]\) is the set of terms \( \langle r_1, \ldots, r_n \rangle \) for all \( r_1 \in [t_1], \ldots, r_n \in [t_n] \).

This definition is extended to an arbitrary ground pool \( P \); \([P]\) is a finite set of precomputed tuples:

- if \( P \) is a tuple \( t_1, \ldots, t_n \) of terms \( (n \neq 1) \) then \([P]\) is the set of tuples \( r_1, \ldots, r_n \) for all \( r_1 \in [t_1], \ldots, r_n \in [t_n] \);
- if \( P \) is a pool \( t_1; \ldots; t_n \) \( (n > 1) \) then \([P]\) is \([t_1] \cup \ldots \cup [t_n] \).

For instance, \([\ldots; \bar{\pi}; \bar{T}; \bar{\pi}]\) is the set \( \{i, j : 1 \leq i, j \leq n\} \).

It is clear that if a ground term \( t \) contains neither symbolic constants nor the symbols \( \langle \) and \( \rangle \) then every element of \([t]\) is a numeral. If a tuple \( t \) of ground terms is precomputed then \([t]\) is \( \{t\} \). The set \([t]\) can be empty. For example, \([1..0] = [1/0] = [1 + a] = \emptyset \).

About a tuple of terms that does not contain .. we say that it is interval-free. It is clear that if a tuple \( t \) of ground terms is interval-free then the cardinality of the set \([t]\) is at most 1.

4.3 Semantics of Arithmetic and Symbolic Literals

For any ground (symbolic or arithmetic) literal \( L \) we will define two translations, \( \tau_\land L \) and \( \tau_\lor L \).

The specific translation function applied to an occurrence of a symbolic or arithmetic literal in a rule depends on the context, as we will see in the following sections.

We will first consider symbolic literals. For any ground atom \( A \),

- if \( A = p(P) \) then \( \tau_\land A \) is the conjunction of atoms \( p(t) \) over all tuples \( t \) in \([P]\), and \( \tau_\lor A \) is the disjunction of these atoms;
• if \( A \) is \( \tilde{p}(P) \) then \( \tau \land A \) is the conjunction of atoms \( \tilde{p}(t) \) over all tuples \( t \) in \([P]\), and \( \tau \lor A \) is the disjunction of these atoms;

• \( \tau \land (\text{not } A) \) is \( \neg \tau \lor A \), and \( \tau \lor (\text{not } A) \) is \( \neg \tau \land A \);

• \( \tau \land (\text{not not } A) \) is \( \neg \neg \tau \land A \), and \( \tau \lor (\text{not not } A) \) is \( \neg \neg \tau \lor A \).

The definitions of \( \tau \land \) and \( \tau \lor \) for arithmetic literals are as follows:

• \( \tau \land (t_1 \prec t_2) \) is \( \top \) if the relation \( \prec \) holds between the terms \( r_1 \) and \( r_2 \) for all \( r_1 \in [t_1] \) and \( r_2 \in [t_2] \), and \( \bot \) otherwise;

• \( \tau \lor (t_1 \prec t_2) \) is \( \top \) if the relation \( \prec \) holds between the terms \( r_1 \) and \( r_2 \) for some \( r_1, r_2 \) such that \( r_1 \in [t_1] \) and \( r_2 \in [t_2] \), and \( \bot \) otherwise.

For instance, \( \tau \lor p(\overline{2..4}) \) is \( p(2) \lor p(3) \lor p(4) \), and \( \tau \lor (\overline{2} \neq \overline{2..4}) \) is \( \top \).

For any tuple \( L \) of ground literals, \( \tau \lor L \) stands for the conjunction of the formulas \( \tau \lor L \) for all members \( L \) of \( L \). The expressions \( \tau \land \bot \) and \( \tau \lor \bot \) both stand for \( \bot \).

It is clear that if \( A \) has the form \( p(t) \) or \( \tilde{p}(t) \), where \( t \) is a tuple of precomputed terms, then each of the formulas \( \tau \land A \) and \( \tau \lor A \) is \( A \).

### 4.4 Semantics of Choice Expressions

The result of applying \( \tau \) to a choice expression \( \{p(P)\} \) is the conjunction of the formulas \( p(t) \lor \neg p(t) \) over all tuples \( t \) in \([P]\). Similarly, the result of applying \( \tau \) to a choice expression \( \{\tilde{p}(P)\} \) is the conjunction of the formulas \( \tilde{p}(t) \lor \neg \tilde{p}(t) \) over all tuples \( t \) in \([P]\).

For instance, the result of applying \( \tau \) to rule \( R_1 \) (see Section 2.3) is

\[
\bigwedge_{1 \leq i, j \leq n} (q(\overline{i, j}) \lor \neg q(\overline{i, j})) .
\]

### 4.5 Global Variables

About a variable we say that it is *global*

• in a conditional literal \( H : L \), if it occurs in \( H \) but does not occur in \( L \);

• in an aggregate literal \( A, \text{not } A \), or \( \text{not not } A \), where \( A \) is of one of the forms (9)–(11), if it occurs in \( s, s_1 \), or \( s_2 \);

• in a rule (12), if it is global in at least one of the expressions \( H_i, B_j \);

• in a rule (13), if it occurs in \( C \) or is global in at least one of the expressions \( B_j \).
An instance of a rule $R$ is any rule that can be obtained from $R$ by substituting precomputed terms for all global variables. A literal or a rule is closed if it has no global variables. It is clear that any instance of a rule is closed.

For example, $X$ is global in the rule $R_2$ from Section 2.3, so that the instances of $R_2$ are rules of the form
\[ \leftarrow r = \bar{1}..\bar{n} \land \text{not count}\{Y : q(r,Y)\} = \bar{1} \]
for all precomputed terms $r$. The variables $X$ and $Y$ are global in $R_4$; instances of $R_4$ are
\[ dI(r,s,r-s+\bar{n}) \leftarrow r = \bar{1}..\bar{n} \land s = \bar{1}..\bar{n} \]
for all precomputed terms $r$ and $s$.

4.6 Semantics of Conditional Literals

If $t$ is a term, $x$ is a tuple of distinct variables, and $r$ is a tuple of terms of the same length as $x$, then the term obtained from $t$ by substituting $r$ for $x$ will be denoted by $t^x_r$. Similar notation will be used for the result of substituting $r$ for $x$ in expressions of other kinds, such as literals and tuples of literals.

The result of applying $\tau$ to a closed conditional literal $H : L$ is the conjunction of the formulas
\[ \tau^x_r(L) \rightarrow \tau^x_r(H) \]
where $x$ is the list of variables occurring in $H : L$, over all tuples $r$ of precomputed terms of the same length as $x$.

For instance, the result of applying $\tau$ to the arithmetic literal $r = \bar{1}..\bar{n}$, where $r$ is a precomputed term, is $\tau^x_r(\epsilon) \rightarrow \tau^x_r(r = \bar{1}..\bar{n})$, where $\epsilon$ is the tuple of length 0. The antecedent of this implication is $\top$. The consequent is $\top$ if $r$ is one of the numerals $\bar{1}, \ldots, \bar{n}$ and $\bot$ otherwise.

4.7 Semantics of Aggregate Literals

In this section, the semantics of ground aggregates proposed by Ferraris [2005] is adapted to closed aggregate literals. Let $E$ be a closed aggregate atom of one of the forms (9)--(11), and let $x_i$ be the list of variables occurring in $t_i : L_i$ ($1 \leq i \leq n$). By $A_i$ we denote the set of tuples $r$ of precomputed terms of the same length as $x_i$. By $A$ we denote the set $\{(i,r) : i \in \{1, \ldots, n\}, r \in A_i\}$.

Let $\Delta$ be a subset of $A$. Then by $[\Delta]$ we denote the union of the sets $[(t_i)^x_r]$ for all pairs $(i,r) \in \Delta$. We say that $\Delta$ justifies $E$ if
- $E$ is of the form (9) and the relation $\prec$ holds between $\gamma[\Delta]$ and $t$ for at least one precomputed term $t \in [s]$, or

---

[13] This definition differs slightly from that given by Harrison et al. [2014]. There, substitutions that yield symbolic constants in the scope of arithmetical operators do not form instances. In a similar way, we treat variables in conditional literals and aggregate literals (Sections 4.6 and 4.7) differently than how they are treated by Harrison et al. [2014].
• $E$ is of the form (10) and the relation $\prec$ holds between $t$ and $\alpha[\Delta]$ for at least one precomputed term $t \in [s]$, or

• $E$ is of the form (11), the relation $\prec_1$ holds between $t_1$ and $\alpha[\Delta]$ for at least one precomputed term $t_1 \in [s_1]$, and the relation $\prec_2$ holds between $\alpha[\Delta]$ and $t_2$ for at least one precomputed term $t_2 \in [s_2]$.

We define $\tau_E$ as the conjunction of the implications

$$
\bigwedge_{(i,r)\in \Delta} \tau_v((L_i)_r^{X_i}) \rightarrow \bigvee_{(i,r)\in A\setminus \Delta} \tau_v((L_i)_r^{X_i})
$$

(22)

over all sets $\Delta$ that do not justify $E$.

For instance, if $E$ is $\text{count}\{p(X) : p(X)\} > 0$ then $\tau E$ is the implication expressing that $p(r)$ holds for at least one precomputed term $r$:

$$
\top \rightarrow \bigvee_r p(r).
$$

If $L$ is an aggregate literal of the form $\text{not} \ E$ where $E$ is an aggregate atom then $\tau L$ is $\neg \tau E$; if $L$ is of the form $\text{not not} \ E$ then $\tau L$ is $\neg \neg \tau E$.

4.8 Semantics of Rules and Programs

For any rule $R$ of form (12), $\tau R$ stands for the set of the formulas

$$
\tau B_1 \land \cdots \land \tau B_m \rightarrow \tau H_1 \lor \cdots \lor \tau H_k
$$

for all instances (12) of $R$. For a rule of form (13), $\tau R$ stands for the set of the formulas

$$
\tau B_1 \land \cdots \land \tau B_m \rightarrow \tau C
$$

for all instances (13) of $R$. For any program $\Pi$, $\tau \Pi$ stands for the union of the sets $\tau R$ for all rules $R$ of $\Pi$.

A stable model of a program $\Pi$ is any stable model of $\tau \Pi$ (in the sense of Section 4.1) that does not contain any pair of atoms of the form $p(t), \bar{p}(t)$.

References


