We will now define the concept of a stable model for sets of formulas that consist of atoms and implications of the form

\[ A_0 \leftarrow A_1 \land \cdots \land A_m \land \neg A_{m+1} \land \cdots \land \neg A_n \]  

(\( n \geq m \geq 1 \)), where each \( A_i \) is an atom.

Let \( \Gamma \) be such a set of formulas, and let \( S \) be a set of atoms. The reduct of \( \Gamma \) relative to \( S \) is the set obtained from \( \Gamma \) by replacing each conjunctive term \( \neg A_i \) in each of the formulas (1) by \( \top \) if \( S \) satisfies \( \neg A_i \), and by \( \bot \) otherwise. The reduct of \( \Gamma \) relative to \( S \) is denoted by \( \Gamma^S \). Since \( \Gamma^S \) is equivalent to a set of definite formulas, it has a unique minimal model. If this model coincides with \( S \) then we say that \( S \) is a stable model of \( \Gamma \).

For instance, let \( \Gamma \) be the pair of formulas

(2) \[ q, \]

(3) \[ p \leftarrow q \land \neg r \]

used as an example in Part 6 of these lecture notes. To check that \( \{p, q\} \) is a stable model of this set, consider the reduct \( \Gamma^{\{p,q\}} \):

\[ q, \]

\[ p \leftarrow q \land \top. \]

The minimal model of the reduct is \( \{p, q\} \), which is exactly the set that we started with.

On the other hand, \( \{q, r\} \) is not a stable model of \( \Gamma \). Indeed, the reduct \( \Gamma^{\{q,r\}} \) is

\[ q, \]

\[ p \leftarrow q \land \bot, \]

and its minimal model is \( \{q\} \)—a set different from \( \{q, r\} \).

7.1. (a) Check that the set of formulas (2), (3) has no stable models other than \( \{p, q\} \). (b) In Part 6 we argued that after adding the formula \( r \leftarrow q \) to this set its stable model becomes \( \{q, r\} \). Prove that that claim was correct.

7.2. Check whether your guesses about the stable models in Problem 6.3 were correct.
The set of formulas

\[ p \leftarrow \neg q, \quad (4) \]
\[ q \leftarrow \neg p \quad (5) \]

has two stable models: \( \{p\} \) and \( \{q\} \). Indeed, the reduct of this set relative to \( \{p\} \) is

\[ p \leftarrow \top, \]
\[ q \leftarrow \bot; \]

its minimal model is \( \{p\} \). Similarly, the reduct of this set relative to \( \{q\} \) is

\[ p \leftarrow \bot, \]
\[ q \leftarrow \top; \]

its minimal model is \( \{q\} \).

7.3. Show that \( \emptyset \) and \( \{p, q\} \) are not stable models of (4), (5).

7.4. Find the stable models of

(a)

\[ p \leftarrow \neg p; \]

(b)

\[ p \leftarrow \neg q, \]
\[ q \leftarrow \neg p, \]
\[ r \leftarrow p, \]
\[ r \leftarrow q. \]