

## Introduction to Mathematical Logic, Handout 5

### Soundness and Completeness of Natural Deduction

A sequent is called *tautological* if

- it has the form  $\Gamma \Rightarrow F$  and  $\Gamma$  entails  $F$ , or
- it has the form  $\Gamma \Rightarrow$  and  $\Gamma$  is unsatisfiable.

An inference rule is said to be *sound* if, for any instance

$$\frac{S_1 \dots S_k}{S}$$

of this rule such that the premises  $S_1, \dots, S_k$  are tautological, the conclusion  $S$  is tautological also.

**Problem 5.1** Rules  $(\rightarrow I)$  and  $(\neg I)$  are sound.

**Problem 5.2** Rule  $(\vee E)$  is sound.

Since all rules of natural deduction are sound, and all axioms are tautological, we can conclude that every sequent that can be proved by natural deduction is tautological. In this sense, the system of natural deduction is sound.

**Problem 5.3** Let  $F$  be a formula,  $A_1, \dots, A_n$  the atoms occurring in  $F$ , and  $I$  an interpretation. Define the literals  $L_1, \dots, L_n$  as follows:

$$L_i = \begin{cases} A_i, & \text{if } I \models A_i, \\ \neg A_i, & \text{otherwise.} \end{cases}$$

Show that

- (a) if  $I \models F$  then the sequent  $L_1, \dots, L_n \Rightarrow F$  can be proved by natural deduction;
- (b) if  $I \not\models F$  then the sequent  $L_1, \dots, L_n \Rightarrow \neg F$  can be proved by natural deduction.

**Problem 5.4** For any tautology  $F$ , the sequent  $\Rightarrow F$  can be proved by natural deduction.

**Problem 5.5** Every tautological sequent can be proved by natural deduction.

In this sense, the system of natural deduction is complete.