Introduction to Mathematical Logic, Handout 7
Predicate Formulas: Semantics

The semantics of propositional formulas described in Handout 2 defines which truth value F^I is assigned to a propositional formula F by an interpretation I. Our goal is to extend this definition to predicate formulas.

First we need to extend the definition of an interpretation to predicate signatures. An *interpretation* I of a predicate signature σ consists of

- a non-empty set $|I|$, called the *universe* of I,
- for every object constant c of σ, an element c^I of $|I|$,
- for every predicate constant P of σ, a function P^I from $|I|^n$ to $\{\mathsf{f},\mathsf{t}\}$, where n is the arity of P.

For instance, the sentence preceding Problem 7.1 can be viewed as the definition of an interpretation of the signature

$$\{a, \mathit{Sum}, \mathit{Prod}\}. \tag{1}$$

For this interpretation I,

$$\begin{aligned}
&|I| = \mathbf{N},\\
&a^I = 0,\\
&\mathit{Sum}^I(\xi,\eta,\zeta) = \begin{cases} \mathsf{t}, & \text{if } \xi+\eta=\zeta,\\ \mathsf{f}, & \text{otherwise,}\end{cases}\\
&\mathit{Prod}^I(\xi,\eta,\zeta) = \begin{cases} \mathsf{t}, & \text{if } \xi\eta=\zeta,\\ \mathsf{f}, & \text{otherwise}\end{cases}
\end{aligned} \tag{2}$$

($\xi,\eta,\zeta \in \mathbf{N}$).

The definition of F^I refers to the substitution operator, defined as follows. Let F be a formula and v a variable. The result of the *substitution* of a term t for v in F, denoted by F^v_t, is the formula obtained from F by replacing each free occurrence of v by t.

Consider an interpretation I of a predicate signature σ. For any element ξ of its universe $|I|$, select a new symbol ξ^*, called the *name* of ξ. By σ^I we denote the predicate signature obtained from σ by adding all names

ξ^* as additional object constants. The interpretation I can be extended to the new signature σ^I by defining

$$(\xi^*)^I = \xi$$

for all $\xi \in |I|$. We will denote this interpretation of σ^I by the same symbol I.

We define the truth value F^I that is *assigned* to F by I for every sentence F of the extended signature σ^I recursively, as follows:

- $P(t_1, \ldots, t_n)^I = P^I(t_1^I, \ldots, t_n^I)$,
- $(\neg F)^I = \neg(F^I)$,
- $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$,
- $(\forall v F)^I = \mathsf{t}$ iff, for all $\xi \in |I|$, $\left(F^v_{\xi^*}\right)^I = \mathsf{t}$,
- $(\exists v F)^I = \mathsf{t}$ iff, for some $\xi \in |I|$, $\left(F^v_{\xi^*}\right)^I = \mathsf{t}$.

As in propositional logic, we say that I *satisfies* F, and write $I \models F$, if $F^I = \mathsf{t}$.

Problem 7.1 Determine which of the sentences

(i) $\exists x Sum(x, x, x)$,

(ii) $\exists x \neg Sum(x, x, x)$,

(iii) $\forall x(Sum(x, x, x) \rightarrow \forall y Sum(x, y, y))$

are satisfied by interpretation (2).

Problem 7.2 Consider the signature consisting of just one symbol, the ternary predicate constant *Sum*. (a) Determine which of the sentences from Problem 7.1 are satisfied by the interpretation I of this signature defined by

$$\begin{aligned} &|I| = \{\mathsf{f}, \mathsf{t}\}, \\ &Sum^I(\xi, \eta, \zeta) = \begin{cases} \mathsf{t}, & \text{if } \vee(\xi, \eta) = \zeta, \\ \mathsf{f}, & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

(b) Determine whether interpretation (3) satisfies the sentence

$$\forall xy(Sum(x, y, x) \vee Sum(x, y, y)).$$

A sentence is *logically valid* if it is satisfied by all interpretations.

Problem 7.3 Determine whether the sentences

$$\exists x(P(x) \wedge Q(x)) \to (\exists x P(x) \wedge \exists x Q(x)),$$
$$(\exists x P(x) \wedge \exists x Q(x)) \to \exists x(P(x) \wedge Q(x))$$

are logically valid.

The *universal closure* of a formula F is the sentence $\forall v_1 \cdots v_n F$, where $v_1, \dots, v_n$ are all free variables of F. About a formula with free variables we say that it is *logically valid* if its universal closure is logically valid.

Problem 7.4 For each of the formulas

$$P(x) \to \exists x P(x),$$
$$P(x) \to \forall x P(x)$$

determine whether it is logically valid.

A formula F is *equivalent* to a formula G if the formula $F \leftrightarrow G$ is logically valid.

Problem 7.5 Determine whether the formula

$$\forall x \exists y P(x, y)$$

is equivalent to

$$\exists y \forall x P(x, y).$$

We say that a set Γ of sentences *entails* a sentence F, or that F is a *logical consequence* of Γ, if every interpretation that satisfies all sentences in Γ satisfies F.

Problem 7.6 Determine whether

$$\exists x P(x, x)$$

is a logical consequence of the sentences

$$\forall x \exists y P(x, y),\ \forall xyz((P(x, y) \wedge P(y, z)) \to P(x, z)).$$