CS313K: Logic, Sets and Functions
Fall 2010

Problem Set 2: Proofs by Induction

Induction is a useful proof method in mathematics and computer science. When we want to prove by induction that some statement containing a variable n is true for all nonnegative values of n, we do two things. First we prove the statement when $n = 0$; this part of the proof is called the *basis*. Then we prove the statement for $n + 1$ assuming that it is true for n; this part of the proof is called the *induction step*. (The assumption that the statement is true for n, which is used in the induction step, is called the *induction hypothesis*.)

Once we have completed both the basis and the induction step, we can conclude that the statement holds for all nonnegative values of n. Indeed, according to the basis, in holds for $n = 0$. From this fact, according to the induction step, we can conclude that it holds for $n = 1$. From this fact, according to the induction step, we can conclude that it holds for $n = 2$. And so on.

Here is an example of the use of induction.

Problem. Prove that for all nonnegative integers n

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Solution. *Basis.* When $n = 0$, the formula turns into

$$0 = \frac{0(0+1)}{2},$$

which is correct. *Induction step.* Assume that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

We need to prove that

$$1 + 2 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{2}.$$

Using the induction hypothesis, we calculate:

$$\begin{aligned} 1+2+\ldots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

2.1. Prove by induction that for all nonnegative integers n

$$1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

2.2. Prove by induction that for all nonnegative integers n

$$1^3+2^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}.$$

2.3. Prove by induction that for all nonnegative integers n

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}.$$

So far we used induction to prove *equalities*. Induction can be also used to prove *inequalities*. Here is an example.

Problem. Prove that for all nonnegative integers n, $2^n > n$.

Solution. *Basis.* When $n = 0$, the formula turns into $1 > 0$, which is correct. *Induction step.* Assume that $2^n > n$. We need to prove that $2^{n+1} > n+1$. This can be done as follows, using the induction hypothesis and then the fact that $2^n \geq 1$:

$$2^{n+1} = 2^n + 2^n > n + 2^n \geq n+1.$$

So far we used induction to prove statements about *nonnegative* integers. Statements about *positive* integers can be proved by induction in a similar way, except that the basis corresponds to $n = 1$; also, in the induction step we may assume that $n \geq 1$. Similarly, if we want to prove a statement about all integers beginning with 2 then the basis corresponds to $n = 2$, and so on.

2.4. Prove that for all integers n beginning with 10, $2^n > n + 1000$.

2.5. Guess which nonnegative values of n satisfy the inequality $3^n > 2^n + n$. Prove your conjecture.

Here is yet another example of the use of induction.

Problem. Prove that for all nonnegative integers n, $n^3 - n$ is a multiple of 3.

Solution. *Basis.* When $n = 0$, we need to check that $0^3 - 0$ is a multiple of 3, which is correct. *Induction step.* Assume that $n^3 - n$ is a multiple of 3. We need to prove that $(n+1)^3 - (n+1)$ is a multiple of 3. This expression can be rewritten as follows:

$$\begin{aligned}(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1) &= n^3 + 3n^2 + 2n \\ &= (n^3 - n) + 3n^2 + 3n.\end{aligned}$$

Consider the three summands $n^3 - n$, $3n^2$, $3n$. By the induction hypothesis, the first of them is a multiple of 3. It is clear that the other two are multiples of 3 also. Consequently, the sum is a multiple of 3.

2.6. Prove that for all nonnegative integers n, $4^n - 1$ is a multiple of 3.