Translation Synchronization via Truncated Least Squares
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Abstract
- 1D Translation Synchronization problem: recover global coordinates \(\{x_i\}\) from noisy relative measurements \(t_{ij} = x_i - x_j + \text{noise}\).
- We introduce a robust algorithm TranSync for this problem by applying truncated least squares to gradually prune out noisy measurements.
- We provide analysis of our algorithm under both deterministic and randomized settings.
- In experiments, our algorithm achieved superior robustness and stability against state-of-the-art convex formulations on both synthetic and real datasets.

Formulations
- \(l_1\) minimization:
  \[
  \arg\min_x \sum_{i,j \in E} |t_{ij} - (x_i - x_j)|
  \]
- Least Squares:
  \[
  \arg\min_x \sum_{i,j \in E} |t_{ij} - (x_i - x_j)|^2
  \]
- Truncated Least Squares (ours):
  \[
  \arg\min_x \sum_{i,j \in E} w^{(k)}_{ij} |t_{ij} - (x_i - x_j)|^2
  \]

Our Algorithm
1. \(x^{(-1)} \leftarrow 0, \delta_{k+1} \leftarrow \infty, c < 1\).
   for \(k = 0, 1, 2, \ldots, k_{\text{max}}\) do
2. Obtain the truncated graph \(G^{(k)}\) using \(x^{(k-1)}\) and \(\delta_{k-1}\).
3. Break if \(G^{(k)}\) is disconnected
4. Solve Truncated Least Squares w.r.t. \(G^{(k)}\) to obtain \(x^{(k)}\).
5. \(\delta_k = \min \{ \max_{i,j \in E} |t_{ij} - (x^{(0)}_i - x^{(0)}_j)|, c\delta_{k-1}\} \).
end for
Output: \(x^{(k)}\)

Deterministic Exact Recovery Condition
- Biased Noise Model (Unbounded Outliers):
  \[
  t_{ij} = \begin{cases} 
  x^{(0)}_i - x^{(0)}_j + U[-\sigma, \sigma] & \text{with probability } p \\
  x^{(0)}_i - x^{(0)}_j + U[-a, b] & \text{with probability } 1 - p 
  \end{cases}
  \]
  \(1\)
- For some constants \(p, q\) only depend on graph structure, during optimization we have
  \[
  \|x^{(k)} - x^{(0)}\|_\infty \leq q\sigma + 2pce^{k-1}
  \]
  and eventually we’ll reach \(\hat{x}\)
  \[
  \|\hat{x} - x^{(0)}\|_\infty \leq \frac{2p + cq}{c - 4p}
  \]
  where the RHS is independent of \(\epsilon\).

Randomized Case
Biased Noise Model:
\[
\begin{align*}
  t_{ij} &= \begin{cases} 
  x^{(0)}_i - x^{(0)}_j + U[-\sigma, \sigma] & \text{with probability } p \\
  x^{(0)}_i - x^{(0)}_j + U[-a, b] & \text{with probability } 1 - p 
  \end{cases} \\
  \end{align*}
\]

Theorem
There exists a constant \(c\) so that if \(p > c/\sqrt{\log(n)}\), then w.h.p.,
\[
\|x^{(k)} - x^{(0)}\|_\infty \leq (1 - p/2)^k (b - a),
\]
\(\forall k = 0, \ldots, \left\lceil \log(1 - p/2) / (b - a) \right\rceil \).

Experiments

Synthetic Graphs in \{Dense, Sparse\} \times \{Regular, Irregular\}

Experiments
- Deterministic Exact Recovery Condition
- Biased Noise Model: 
  \[
  t_{ij} = \begin{cases} 
  x^{(0)}_i - x^{(0)}_j + U[-\sigma, \sigma] & \text{with probability } p \\
  x^{(0)}_i - x^{(0)}_j + U[-a, b] & \text{with probability } 1 - p
  \end{cases}
  \]

Table: Global ranking of selected six movies via different methods: MRQE, HodgeRank[1] with 1) arithmetic mean score difference, 2) geometric mean score ratio and 3) binary comparisons, and the initial and final predictions of TranSync. TranSync results in the most consistent result with MRQE.

Ranking from relative comparisons.

Table:  Global ranking (score)

Reference
- Xiaoye Jiang, Lek-Heng Lim, Yuan Yao, and Yinyu Ye.
  Statistical ranking and combinatorial hodge theory.