Practice Final Examination Solutions

CS 313H

1. [10] Use a truth table to determine for which truth values of \( p, q, \) and \( r \) \( \sim (p \land r) \lor (\sim q \land r) \) is true.

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<tr>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( p \land r )</th>
<th>( \sim (p \land r) )</th>
<th>( \sim q )</th>
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<th>( \sim (p \land r) \lor (\sim q \land r) )</th>
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The expression \( \sim (p \land r) \lor (\sim q \land r) \) is true for all truth values of \( p, q, \) and \( r \) except all of \( p, q, \) and \( r \) being true.

2. [20] Using sentential calculus (with a four column format), prove that the conclusion \( p \) follows from premises: \( p \lor q, q \Rightarrow t, \sim r \lor \sim s, (s \land t) \Rightarrow r, \) and \( q \Rightarrow s . \)

\[
\begin{align*}
\{P_r_1\} & \quad (1.) p \lor q \quad P \\
\{P_r_2\} & \quad (2.) q \Rightarrow t \quad P \\
\{P_r_3\} & \quad (3.) \sim r \lor \sim s \quad P \\
\{P_r_4\} & \quad (4.) (s \land t) \Rightarrow r \quad P \\
\{P_r_5\} & \quad (5.) q \Rightarrow s \quad P \\
\{P_r_6\} & \quad (6.) \sim p \quad P \text{ (for CP)} \\
\{P_r_1, P_r_2\} & \quad (7.) q \quad DS, (1), (6) \\
\{P_r_1, P_r_2, P_r_3\} & \quad (8.) t \quad MP (2), (7) \\
\{P_r_1, P_r_2, P_r_3, P_r_5\} & \quad (9.) s \quad MP (5), (7) \\
\{P_r_1, P_r_2, P_r_3, P_r_5, P_r_6\} & \quad (10.) s \land t \quad \text{Conj. (8), (9)} \\
\{P_r_1, P_r_2, P_r_3, P_r_5, P_r_6\} & \quad (11.) r \quad MP (4), (10) \\
\{P_r_1, P_r_2, P_r_3, P_r_5, P_r_6\} & \quad (12.) \sim s \quad DS, (3), (11) \\
\{P_r_1, P_r_2, P_r_3, P_r_5, P_r_6\} & \quad (13.) p \quad \text{ContraPrm. (9), (12)} \\
\{P_r_1, P_r_2, P_r_3, P_r_5\} & \quad (14.) \sim p \Rightarrow p \quad C (6), (13) \\
\{P_r_1, P_r_2, P_r_3, P_r_5\} & \quad (15.) p \quad \text{Clav (6), (13)}
\end{align*}
\]
3. [20] Prove that the conclusion $p$ follows from the premises $((p \Rightarrow q) \land (p \land \neg q)) \lor r$ and $r \Rightarrow p$. First convert the premises and the negation of the conclusion into Conjunctive Normal Form, and then employ a resolution proof to get a contradiction.

\begin{align*}
((p \Rightarrow q) \land (p \land \neg q)) \lor r \\
((\neg p \lor q) \land (p \land \neg q)) \lor r \\
((\neg p \lor q \lor r) \land ((p \land \neg q) \lor r)) \\
(\neg p \lor q \lor r) \land (p \lor r) \land (\neg q \lor r)
\end{align*}

$r \Rightarrow p$

$\neg r \lor p$

$\neg p$

1. $\neg p \lor q \lor r$ P
2. $p \lor r$ P
3. $\neg q \lor r$ P
4. $\neg r \lor p$ P
5. $\neg p$ P
6. $\neg r$ Res (4), (5)
7. $p$ Res (2), (6)
8. false Conj. (5), (7)

4. [10] Using the predicates defined on the set $L$ of upper case Latin characters, the set $\mathbb{N}^+$ of positive integers, and the set $S$ of strings of upper case Latin characters:

- $Vx$ $x$ is a vowel, for $x \in L$
- $Sxn$ $x$ can be written in $n$ strokes, for $x \in L$ and $n \in \mathbb{N}^+$
- $Wxs$ $x$ occurs in the string $s$, for $x \in L$ and $s \in S$
- $Bxy$ $x$ occurs before $y$ in the English alphabet, for $x, y \in L$
- $Exy$ $x$ equals $y$, for $x, y \in L$

Express in the syntax of Predicate Calculus (you may use upper case Latin characters, positive integers, and strings of upper case Latin characters as constants):

- a. ‘A’ is the only upper case Latin character that is a vowel and can be written in three strokes but does not occur in the string ‘STUPID’.

\[(\forall x \in L)((Vx \land Sx3 \land \neg Wx 'STUPID') \iff (ExA))\]

- b. There is an upper case Latin character strictly between ‘K’ and ‘R’ that can be written in one stroke.

\[(\exists x \in L)(BKx \land BxR \land Sx1)\]
5. [25] Prove that $(\forall u)(\exists v)Rvu$ follows from $(\exists x)(\forall y)Rxy$ (Rather than using the TC rule be specific about the sentential calculus rule.)

\[
\begin{array}{ll}
\{P_1\} & (1). \ (\exists x)(\forall y)Rxy \\
\{P_1\} & (2). \ (\forall y)Ray \\
\{P_1\} & (3). \ Rab \\
\{P_1\} & (4). \ (\exists y)Rvb \\
\{P_1\} & (5). \ (\forall u)(\exists v)Rvu \\
\end{array}
\]

6. [10] Using induction, prove that for \( n \geq 1 \), \( \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 \).

For \( n \geq 1 \), let \( P(n) = \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 \). 

Basis step: \( P(1) \) is true since \( \sum_{k=1}^{1} k \cdot k! = 1 \cdot 1! = 1 = 2 - 1 = (1+1)! - 1 \).

Inductive step: For \( n \geq 1 \), \( P(n) \Rightarrow P(n+1) \), since if \( \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 \), then 
\[
\sum_{k=1}^{n+1} k \cdot k! = \sum_{k=1}^{n} k \cdot k! + (n+1)(n+1)! \\
= (n+1)! - 1 + (n+1)(n+1)! \\
= (n+1+1)(n+1)! - 1.
\]

7. [10] a. Given a sequence of integers \( a_1, a_2, \ldots \) such that \( a_k > a_{k-1} \) for \( k \geq 2 \), using induction to prove that for \( k \geq 1 \), \( a_k \geq a_1 + k - 1 \). (Notice \( a_k > a_{k-1} \) is equivalent to \( a_k \geq a_{k-1} + 1 \).)

For \( k \geq 1 \), let \( P(k) = \ \ \ \ a_k \geq a_1 + k - 1 \).

Basis step: \( P(1) \) is true since \( a_1 \geq a_1 + (1-1) \).

Inductive step: For \( k \geq 1 \), \( P(k) \Rightarrow P(k+1) \), since if \( a_k \geq a_1 + k - 1 \), then 
\[
a_{k+1} = (a_{k+1} - a_k) + a_k \\
\geq a_1 + k - 1 + (a_{k+1} - a_k) \\
\geq a_1 + k - 1 + 1 \\
\geq a_1 + (k+1) - 1.
\]

b. [5] Using this, prove that for any integer \( m \), display a \( k \) so that \( a_k \geq m \).

Given any integer \( m \), let \( k = \max \{1, m+1-a_1\} \) then \( k \geq 1 \) and \( k \geq m+1-a_1 \) so 
\[
a_k \geq a_1 + k - 1 \geq a_1 + m + 1 - a_1 - 1 = m.
\]
8. [10] Prove for any sets $A, B,$ and $C$ that $A \sim (B \sim C) = (A \sim B) \cup (A \cap C)$.

We have

\[
x \in A \sim (B \sim C) \\
\Leftrightarrow x \in A \land \sim (x \in B \sim C) \\
\Leftrightarrow x \in A \land \sim (x \in B \land x \notin C) \\
\Leftrightarrow x \in A \land (x \notin B \lor x \in C) \\
\Leftrightarrow (x \in A \land x \notin B) \lor (x \in A \land x \in C) \\
\Leftrightarrow (x \in A \sim B) \lor (x \in A \cap C) \\
\Leftrightarrow x \in (A \sim B) \lor (A \cap C)
\]

9. [10]. Given sets $A, B, C,$ and $D$ be sets. Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C \land B \subseteq D$.

Suppose $A \subseteq C \land B \subseteq D$, then

\[
(x, y) \in A \times B \\
\Rightarrow x \in A \land y \in B \\
\Rightarrow x \in C \land y \in D \\
\Rightarrow (x, y) \in C \times D.
\]

Suppose $A \times B \subseteq C \times D$, then

\[
x \in A \land y \in B \\
\Rightarrow (x, y) \in A \times B \\
\Rightarrow (x, y) \in C \times D \\
\Rightarrow x \in C \land y \in D,
\]

Thus $A \subseteq C \land B \subseteq D$.

10. [15]. Let $R$ be defined $R = \{(x, y), (u, v) : x^2 + y^2 = u^2 + v^2\}$. Prove that $R$ is an equivalence relation on $\mathbb{R}^2$.

We need to show that $R$ is reflexive, symmetric, and transitive. For any $(x, y) \in \mathbb{R}^2$

\[
x^2 + y^2 = x^2 + y^2\] so $(x, y), (x, y)) \in R$ and $R$ is reflexive. Next since for $(x, y), (u, v) \in \mathbb{R}^2$ if $x^2 + y^2 = u^2 + v^2$ then $u^2 + v^2 = x^2 + y^2$ so $(x, y), (u, v)) \in R$ implies $(u, v), (x, y)) \in R$ is symmetric. Lastly, if $(x, y), (u, v)) \in R$ and $(u, v), (w, z)) \in R$ then

\[
x^2 + y^2 = u^2 + v^2\] and $u^2 + v^2 = w^2 + z^2\] so $x^2 + y^2 = w^2 + z^2\] and $(x, y), (w, z)) \in R$, and $R$ is transitive. We conclude $R$ is an equivalence relation on $\mathbb{R}^2$.

11. Consider a relation $R$ on a set $A$. Prove or disprove with a simple counter example each of the following:
a. [10] If \( R \) is reflexive, then \( R^2 \) is reflexive.

For all \( x \in A \), since \( R \) is reflexive then \((x,x) \in R\). But then \((x,x) \in R\) and \((x,x) \in R^2\) imply \((x,x) \in R^2\) so \( R^2 \) is reflexive.

b. [10] If \( R \) is symmetric, then \( R^2 \) is symmetric.

For all \( x, y \in A \), if \((x, y) \in R^2\) then for some \( z \in A \), \((x, z) \in R\) and \((z, y) \in R\). But then since \( R \) is symmetric for the same \( z \), \((z, x) \in R\) and \((y, z) \in R\) so then \((y, x) \in R^2\) so \( R^2 \) is symmetric.

c. [10] If \( R \) is antisymmetric, then \( R^2 \) is antisymmetric.

Let \( A = \{1,2,3\} \) and \( R = \{(1,2),(2,3),(3,1),(3,3)\} \) so \( R^2 = \{(1,3),(2,3),(2,1),(3,2),(3,1)\} \). \( R \) is antisymmetric, but \( R^2 \) is not antisymmetric since both \((2,3),(3,2) \in R^2\).

d. [10] If \( R \) is transitive, then \( R^2 \) is transitive.

For all \( x, y \in A \), if \((x, y), (y, z) \in R^2\) then for some \( z \in A \), \((x, z) \in R\) and \((z, y) \in R\) and for some \( w \in A \), \((y, w) \in R\) and \((w, z) \in R\). Since \( R \) is transitive \((x, y) \in R\) and \((y, z) \in R\) so then \((x, y) \in R^2\) so \( R^2 \) is transitive.

12. [10] Given \( f : \mathbb{N}^+ \to \mathbb{N}^+ \) and \( g : \mathbb{N}^+ \to \mathbb{N}^+ \) defined by \( f(n) = n^2 \) and \( g(n) = 2^n \), respectively, what are

a. \( f \circ f \)

\( f \circ f(n) = f(f(n)) = (n^2)^2 = n^4 \).

b. \( f \circ g \)

\( f \circ g(n) = f(g(n)) = (2^n)^2 = 2^{2n} \).

c. \( g \circ f \)

\( g \circ f(n) = g(f(n)) = 2^{(n^2)} \).

d. \( g \circ g \)

\( g \circ g(n) = g(g(n)) = 2^{(2^n)} \).

13. [20] Given a sets \( A \) and \( B \) and function \( f : A \to B \), Prove that \( f \) is one-to-one if and only if \( f(X \cap Y) = f(X) \cap f(Y) \) for all \( X, Y \subseteq A \). (Hint: Recall \( f(\emptyset) = \emptyset \) and \( f(\{x\}) = \{f(x)\} \) for any \( x \in A \).)
We will first show that if \( f(X \cap Y) = f(X) \cap f(Y) \) then \( f \) is one-to-one. To that end, suppose \( f(X \cap Y) = f(X) \cap f(Y) \) and \( f(x) = f(y) \). Then \( f([x]) = f([y]) \) and \( \emptyset \neq f([x]) = f([x]) \cap f([y]) = f([x] \cap [y]) \) so \( [x] \cap [y] \) is nonempty, but this means \( x = y \) so \( f \) is one-to-one.

Next we will show that if \( f \) is one-to-one then \( f(X) \cap f(Y) \subseteq f(X \cap Y) \). Take any \( z \in f(X) \cap f(Y) \) thus \( z \in f(X) \) and \( z \in f(Y) \), so for some \( x \in X \), \( f(x) = z \) and for some \( y \in Y \), \( f(y) = z \) . But \( f \) is one-to-one, so \( x = y \) . Since \( x = y \) and \( y \in Y \) we have \( x \in Y \) and \( x \in X \cap Y \) . Since \( f(x) = z, z \in X \cap Y \) and we have proved that \( f(X) \cap f(Y) \subseteq f(X \cap Y) \).

Lastly, we show that \( f(X \cap Y) \subseteq f(X) \cap f(Y) \). (This is actually true whether \( f \) is one-to-one or not.) . Take any \( z \in f(X \cap Y) \) thus for some \( x \in X \cap Y \), \( f(x) = z \) and for some \( y \in Y \), \( f(y) = z \) . Since \( x \in X \cap Y, z \in f(X) \) and \( z \in f(Y) \), so \( z \in f(X) \cap f(Y) \) and \( f(X \cap Y) \subseteq f(X) \cap f(Y) \).