Asymptotic Dominance Problems

1. Display a function \( f : \mathbb{N} \to \mathbb{R} \) that is \( O(1) \) but is not constant.

The function \( f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases} \) is not constant but for \( n \geq 0 \), \( |f(n)| \leq 1 \cdot |n| \).

2. Define the relation "\( \leq \)" on functions from \( \mathbb{N} \) into \( \mathbb{R} \) by \( f \leq g \) if and only if \( f = O(g) \).

Prove that \( \leq \) is reflexive and transitive. (Recall: to be reflexive, you must

To prove reflexivity, notice that for any \( f : \mathbb{N} \to \mathbb{R} \) and all \( n \geq 0 \), \( |f(n)| \leq 1 \cdot |f(n)| \).

To prove transitivity, suppose \( f = O(g) \) and \( g = O(h) \), then by definition, there exist \( N_f \geq 0, M_f \geq 0, N_g \geq 0, M_g \geq 0 \), so that for \( n \geq N_f \), \( |f(n)| \leq M_f |g(n)| \) and for \( n \geq N_g \), \( |g(n)| \leq M_g |h(n)| \). Thus for \( n \geq \max \{N_f, N_g\} \), \( |f(n)| \leq M_f M_g |h(n)| \). We may conclude that \( f = O(h) \).

3. Suppose \( f = O(g) \) and \( g = O(h) \), prove or disprove (with a simple counter-example) that \( f = O(h) \).

Suppose \( f = O(g) \) and \( g = O(h) \), then by definition, there exist

\( N_f \geq 0, M_f \geq 0, N_g \geq 0, M_g \geq 0 \), so that for \( n \geq N_f \), \( |f(n)| \leq M_f |g(n)| \) and for

\( n \geq N_g \), \( |g(n)| \leq M_g |h(n)| \). Thus for \( n \geq \max \{N_f, N_g\} \), \( |f(n)| \leq M_f M_g |h(n)| \). We may conclude that \( f = O(h) \).

4. Suppose \( f = o(g) \) and \( g = O(h) \). Prove that \( f = o(h) \).

Since \( g = O(h) \), there exist \( M_1 \) and \( N_1 \) so that \( n \geq N_1 \Rightarrow |g(n)| \leq M_1 |h(n)| \). Given \( \varepsilon > 0 \), let \( \varepsilon' = \varepsilon / M_1 \). Since \( f = o(g) \), there exist \( N_2 \) such that

\( n \geq N_2 \Rightarrow |f(n)| \leq \varepsilon' |g(n)| = \varepsilon / M_1 |g(n)| \). Thus letting \( N = \max \{N_1, N_2\} \), for

\( n \geq N \) we have \( |f(n)| \leq \varepsilon / M |g(n)| \leq \varepsilon |h(n)| \) so \( f = o(h) \).

5. Suppose \( f = O(g) \) and \( g = O(h) \). If \( h = O(f) \), prove that \( h = O(g) \).

By definition, there exist \( N_f \geq 0, M_f \geq 0, N_h \geq 0, M_h \geq 0 \), so that for \( n \geq N_f \),

\( |f(n)| \leq M_f |g(n)| \) and for \( n \geq N_h \), \( |h(n)| \leq M_h |f(n)| \). Thus for \( n \geq \max \{N_f, N_h\} \),

\( |h(n)| \leq M_f M_h |g(n)| \). We may conclude that \( h = O(g) \).
6. Using Theorem 2 and induction prove that if for \( i = 1, 2, ..., k \), \( f_i = O(g_i) \), then 
\[
\sum_{i=1}^{k} f_i = O(\sum_{i=1}^{k} |g_i|).
\]

For \( k = 1 \), we have \( \sum_{i=1}^{1} f_i = f_1 = O(g_1) = O(\sum_{i=1}^{1} g_i) \). Now assume \( \sum_{i=1}^{k} f_i = O(\sum_{i=1}^{k} |g_i|) \) and consider \( \sum_{i=1}^{k+1} f_i \). Since \( \sum_{i=1}^{k} f_i = O(\sum_{i=1}^{k} |g_i|) \) and \( f_{k+1} = O(g_{k+1}) \), Theorem 2 guarantees that 
\[
\sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1} = O(\sum_{i=1}^{k} |g_i| + |g_{k+1}|) = O(\sum_{i=1}^{k+1} |g_i|).
\]

7. Employing induction and Theorem 3, prove that if for \( i = 1, 2, ..., k \), \( f_i = O(g) \), then 
\[
\sum_{i=1}^{k} f_i = O(g).
\]

For \( k = 1 \), we have \( \sum_{i=1}^{1} f_i = f_1 = O(g) \) by hypothesis. Now assume \( \sum_{i=1}^{k} f_i = O(g) \) and consider \( \sum_{i=1}^{k+1} f_i \). Since \( \sum_{i=1}^{k} f_i = O(g) \) and \( f_{k+1} = O(g) \), Theorem 3 guarantees that 
\[
\sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1} = O(\max\{ |g_i|, |g_{k+1}| \}) = O(\max\{|g_i|}) = O(g).
\]

8. Show that if \( f(n) = 12n + 3 \) and \( g(n) = n^2 \), then \( f = O(g) \).

Let \( N = 3 \) and \( M = 13 \). For \( n \geq N \):

\[
|f(n)| = |12n + 3| = 12n + 3 \leq 12n + n = 13n \leq 13n^2 = 13 |n^2| = M |g(n)|.
\]

Thus \( f = O(g) \).

9. Define \( f : N \to R \) by \( f(n) = \begin{cases} 10^{100} & \text{for } n = 17 \\ n & \text{for } n \neq 17 \end{cases} \). Prove that \( f = O(n) \).

For \( n \geq 18 \), \( |f(n)| = |n| \leq 1 \cdot |n| \), so \( f = O(n) \).
10. Consider the functions $f$ and $g$ defined on $\mathbb{N}$ by 

\[
 f(n) = \begin{cases} 
 n^2 & \text{for } n \text{ even} \\
 2n & \text{for } n \text{ odd}
\end{cases}
\]

and 

\[
 g(n) = n^2. 
\]

Show that $f = O(g)$ but that $f \neq o(g)$ and $g \neq O(f)$.

$f = O(g)$: Since for $n \geq 0$, $2n \leq 2n^2$; we have that $|2n| \leq 2|n^2|$ and $|n^2| \leq 2|n^2|$, so $|f(n)| \leq 2|g(n)|$. Thus $f = O(g)$.

$f \neq o(g)$: Suppose $f = o(g)$, then for $\varepsilon = 1/2$ there is a non-negative $N$ so that for all $n \geq N$, $|f(n)| \leq \varepsilon |g(n)|$. But letting $n = 2$ if $N = 0$ and $n = N$ or $N+1$ (whichever is even) if $N$ is positive, we have $|f(n)| = n^2 > \frac{1}{2}n^2 = \varepsilon |g(n)|$. This is a contradiction, so $f \neq o(g)$.

$g \neq O(f)$: Suppose $g = O(f)$, then there exist nonnegative $M$ and $N$ so that for all $n \geq N$, $|g(n)| \leq M |f(n)|$. But letting $n$ be odd and greater than $N$ and $2M$, then we have $|g(n)| = n^2 = n \cdot n > 2Mn = M \cdot 2n = M |f(n)|$. This is a contradiction, so $g \neq O(f)$.

11. Show that $2^n = O(n!)$. 

For $n \geq 2$ and $i = 2, 3, \ldots, n$, we have $2 \leq i$, thus \( \prod_{i=2}^{n} 2 \leq \prod_{i=2}^{n} i \). Therefore, 

\[
 2^n = \prod_{i=1}^{n} 2 = 2 \cdot \prod_{i=2}^{n} 2 \leq 2 \cdot \prod_{i=2}^{n} i = 2 \cdot \prod_{i=1}^{n} i = 2^n \cdot n! \text{ and we have } |2^n| \leq 2 \cdot |n!|, \text{ thus } 2^n = O(n!). \]

12. Show that for any real value of $a$, $a^n = O(n!)$. (Hint: be careful to consider negative values of $a$.)

Define $K = \lceil a \rceil$ (i.e. $K$ is the first integer greater than or equal to $|a|$). For $n \geq K$ and $i = K, K+1, \ldots, n$, we have $|a| \leq i$, thus \( \prod_{i=K}^{n} |a| \leq \prod_{i=K}^{n} i \). Therefore, 

\[
 |a|^n = \prod_{i=1}^{n} |a| = |a|^{K-1} \cdot \prod_{i=K}^{n} |a| \leq |a|^{K-1} \cdot \prod_{i=K}^{n} i \leq |a|^{K-1} \cdot \prod_{i=1}^{n} i = |a|^{K-1} n!. \] 

So with $M = |a|^{K-1}$ and $N = K$, we have $|a^n| \leq M \cdot |n!|$ for all $n \geq N$. Thus $a^n = O(n!)$. 

13. Show that for any $b > 1$, $\log_b n = o(n)$

Consider any positive $\varepsilon$, and choose $N = \left\lceil 1 + \frac{2}{(b^\varepsilon - 1)^2} \right\rceil$. Then, if $n > N$, we have

$$n > 1 + \frac{2}{(b^\varepsilon - 1)^2}, \text{ thus } \frac{(n-1)}{2} (b^\varepsilon - 1)^2 > 1, \text{ and } \frac{n(n-1)}{2} (b^\varepsilon - 1)^2 > n.$$ But using the binomial theorem, we have

$$b^n = (b^\varepsilon)^n = (1 + (b^\varepsilon - 1))^n = \sum_{j=0}^{n} \binom{n}{j} (b^\varepsilon - 1)^j > \left(\frac{n}{2}\right) (b^\varepsilon - 1)^2 > n.$$ By taking base $b$ logarithms, we have

$$\varepsilon |n| = n = \log_b b^n > \log_b \varepsilon n = \left|\log_b n\right|.$$

14. Prove that if $0 \leq a < b$, then $a^n = o\left(b^n\right)$

If $a = 0$, then for all $\varepsilon > 0$ and all $n \geq 1$, we have $\left|a^n\right| = 0 \leq \varepsilon |b^n|$. Assume now that $a > 0$. Take $N = \ln(\varepsilon) / \ln(a/b)$ and (assuming $\varepsilon < 1$), for $n \geq N$,

$$n \cdot \ln(a/b) \leq \ln(\varepsilon) \text{ and } \left|a^n\right| = a^n \leq \varepsilon \cdot b^n = \varepsilon |b^n|.$$ If $\varepsilon \geq 1$ then

$$\left|a^n\right| = a^n \leq b^n \leq \varepsilon \cdot b^n = \varepsilon |b^n| \text{ for } n \geq 0.$$ Thus $a^n = o\left(b^n\right)$.

15. Prove that if $0 \leq a < b$, then $n^a = o\left(n^b\right)$

Given any $\varepsilon > 0$, let $N = \left(1/\varepsilon\right)^{1/(b-a)}$. Notice then for $n \geq N = \left(1/\varepsilon\right)^{1/(b-a)}$, $n^{b-a} \geq 1/\varepsilon$, and $n^{-(b-a)} \leq \varepsilon$. So $\left|n^a\right| = \left|n^{-(b-a)} n^b\right| = \left|n^{-(b-a)} n^b\right| \leq \varepsilon \left|n^b\right|$. Therefore, $n^a = o\left(n^b\right)$.

16. Prove that if $0 < a < b$, then $b^n \neq O(a^n)$

Given $M \geq 0$ and $N \geq 0$, let $\overline{M} = \max\{M,1\}$ thus $\overline{M} \geq M$ and $\ln(\overline{M}) \geq 0$. Notice that $\ln(\overline{b/a}) > 0$ and choose $n = \max\{N, \left\lceil \ln(\overline{M}) \frac{\ln(b/a)}{\ln(b/a)} \right\rceil + 1$. For this $n$ we have $n > \frac{\ln(\overline{M})}{\ln(b/a)}$, thus $n \ln(b/a) > \ln(\overline{M})$ and $(b/a)^n \geq \overline{M} \geq M$. But then $\left|b^n\right| = b^n \geq M a^n = M \left|a^n\right|$ so $b^n \neq O(a^n)$.
17. Prove that $\sqrt{n} = O(n^2)$.

Let $M = 1$ and $N = 1$. For $n \geq N, n^{3/2} \geq 1$. Thus $|\sqrt{n} - \sqrt{n}n^{3/2}| = n^{3/2} = 1 |n^2|$, so $\sqrt{n} = O(n^3)$.

18. Prove that $e^{(n^2)} \neq o(e^n)$.

Let $\varepsilon = 1$, consider and $N$, and choose $n \geq \max\{N, 2\}$. Since $n \geq 2, n^2 \geq 2n > n$ and $|e^{n^2}| > e^n = e |n|$ so $e^{(n^2)} \neq o(e^n)$.

19. Using only Definition 1, prove that $3n^4 = O(n^{0.5})$.

Let $M = 3$ and $N = 1$. For $n \geq N = 1$, we have $\sqrt{n} \geq 1$, so $|3n^4| \leq 3 n^4 \sqrt{n} = 3 |n^{4.5}|$. Thus $3n^4 = O(n^{0.5})$.

20. Using only Definition 2, prove that $5^n \neq o(2 \cdot 4^n)$.

Let $\varepsilon = 1/4$ and suppose there exists $N$ so that for all $n \geq N$, $|5^n| \leq \varepsilon |2 \cdot 4^n|$. But for $n = \max\{1, [N]\}$, we have $n \geq N$ and $n \geq 1$, so $\left(\frac{5}{4}\right)^n > 1$ and $5^n > 4^n$, thus $|5^n| \leq 5^n > 4^n = 1/2 |2 \cdot 4^n| = \varepsilon |2 \cdot 4^n|$ and $5^n \neq o(2 \cdot 4^n)$.

21. Show that if $f(n) = n^2$ and $g(n) = n$, then $f \neq o(g)$.

Let $\varepsilon = 1$ and consider any positive $N$. Let $n = N + 1$ so $n \geq 2$ and $n \geq N$. We have:

$|f(n)| = |n^2| = |n| \cdot |n| \geq 2 |n| > \varepsilon |n| = \varepsilon |g(n)|$.

Thus $f \neq o(g)$.

22. Show that $\log_2 n! = O(n \log_2 n)$ and $n \log_2 n = O(\log_2 n!)$.

For $n \geq 1$, we have $\log_2 n! = \sum_{i=1}^{n} \log_2 i \leq \sum_{i=1}^{n} \log_2 n = n \log_2 n$. Thus $|\log_2 n!| \leq 1 |n \log n|$ and $\log_2 n! = O(n \log_2 n)$. To show $n \log_2 n = O(\log_2 n!)$ let $N = 8$ and $M = 3$. Notice that if $n \geq 8, \frac{n}{8} \geq 1$, so $\left(\frac{n}{2}\right)^3 = \frac{n^3}{8} \geq n^2 \geq n^2$. Also notice that

$\left[\frac{n}{2}\right] = 1 \leq \frac{n}{2} + 1 = \frac{n - n - \frac{n}{2}}{2} = \frac{n}{2}$. Finally

$n^3 = (n^2)^{n/2} \leq \left(\frac{n}{2}\right)^{3/2} \leq \left(\frac{n}{2}\right)^3 = \prod_{k=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \left(\frac{n}{2}\right)^3 \leq \prod_{k=1}^{\left\lfloor\frac{n}{2}\right\rfloor} k^3 \leq \prod_{k=1}^{\left\lfloor\frac{n}{2}\right\rfloor} k^3 = (n!)^3$. 


By taking logs, we have for $n \geq 8$, $|n \log_2 n| = n \log_2 n \leq 3 \log_2 n! = 3 |\log_2 n!|$. 