1. Present a combinatorial argument that for all positive integers \( x \) and \( y \)
\[
\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x + y)^n.
\]
(Hint: Consider sequences drawn from the union of distinct sets \( A \) and \( B \) of cardinalities \( x \) and \( y \), respectively.)

2. Present a combinatorial argument that for all positive integers \( 1 \leq k \leq m \leq r \):
\[
\binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k}.
\]

3. a. Present a combinatorial argument that for all positive integers \( n, a, \) and \( b(>a) \):
\[
\sum_{k=0}^{n} \binom{n}{k} a^k (b-a)^{n-k} = b^n.
\]

   b. Present a combinatorial argument that for all positive integers \( n \):
\[
\binom{2n}{2} = 2\binom{n}{2} + n^2.
\]

4. Using a combinatorial argument, prove that for \( n \geq 1 \):
\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.
\]

5. a. Present a combinatorial argument that for all positive values of \( n \):
\[
3^n = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j}.
\]

   b. Present a combinatorial argument that for all \( m \) and \( n \) satisfying \( 2 \leq m, 2 \leq n \), and \( m \leq n + 1 \):
\[
\binom{n+2}{m} = \binom{n+1}{m-1} + \binom{n}{m} + \binom{n}{m-2}.
\]
(Hint: Consider \( A = B \cup \{c\} \cup \{d\} \), where \( c \neq d \), \( c \not\in B, d \not\in B \), and \#\( B = n \).)

6. a. Present a combinatorial argument that for all \( n \) and \( k \) satisfying \( 1 \leq n \) and \( k \leq n \):
\[
n! = \binom{n}{k} k! \cdot (n-k)!
\]

   b. Present a combinatorial argument that for all positive values of \( n \):
\[
2^n = 1 + \sum_{k=0}^{n-1} 2^k
\]
(Hint: Consider Let \( k \) be the position of the first 1 in a bit string.)
7. a. Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

b. Present a combinatorial argument that for all nonnegative integers $p, s$, and $n$ satisfying $p + s \leq n$

$$\binom{n}{p-s} = \binom{n}{p+s} \binom{p+s}{p}$$

(Hint: Consider choosing two subsets.)

8. a. Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$$

(Note: The summation begins with $k = 1$.)

b. Present a combinatorial argument that for all integers $k$ and $n$ satisfying $3 \leq k \leq n$

$$\binom{n}{k} = \binom{n-3}{k} + 3 \binom{n-3}{k-1} + 3 \binom{n-3}{k-2}$$

(Hint: Consider three special elements.)

9. Present a combinatorial argument that for all positive integers $m, n,$ and $r$, satisfying $r \leq \min\{m, n\}$:

$$\binom{m+n}{r} = \sum_{k=0}^{n} \binom{m}{k} \binom{n}{r-k}.$$ 

(Hint: Consider selecting from two sets.)

b. Present a combinatorial argument that for all positive integers $n$:

$$3^n = \sum_{i=0}^{n} \left( \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} \right)$$

(Note: Be very specific about the roles of $i$ and $j.$)