1 Problem Statement

Given are a number of coins out of which one is known to defective. It is required to isolate the defective coin by weighing groups of coins using a balance. In all cases the goal is to minimize the number of weighings. It is assumed that there is exactly one defective coin. A reference coin is a good coin. In each of the following cases below, $r$ reference coins are available. The $n$ potentially defective coins are called pdc.

$L(n, r)$: minimum number of weighings required to isolate the defective coin out of $n$ coins given that the defective coin is light.

$H(n, r)$: minimum number of weighings required to isolate the defective coin out of $n$ coins given that the defective coin is heavy.

$P(n, r)$: minimum number of weighings required to isolate the defective coin out of $n$ coins.

$D(n, r)$: minimum number of weighings required to isolate the defective coin out of $n$ coins and diagnose it (i.e., pronounce that the defective coin is light or heavy).

Observations In the following $F$ stands for any of the following functions: $L, H, P, D$.

1. $L(n, r) = H(n, r)$, by symmetry.

2. $L(n, r) \leq P(n, r) \leq D(n, r)$: the first inequality is established by observing that any procedure for isolating a defective coin out of $n$ can be applied to locating the defective coin that is known to be light; the additional information is not used. The second inequality is similar. It follows that $[L(n, r) = D(n, r)] \Rightarrow [L(n, r) = P(n, r) = D(n, r)]$.

3. $F(n, r + 1) \leq F(n, r)$.

4. $F(n, r + k) \leq F(n + k, r)$, $k \geq 0$: A procedure for $(n, r + k)$ can be adapted to $(n + k, r)$ by taking $k$ reference coins and adding them to the set of potentially defective coins.

2 Enumerating Certain Function Values

We write $-$ for $r$ when the value is immaterial, and $+$ when it is any positive number. Since $L = H$, henceforth we work with $L$ only. The following values
are straightforward:

\[ L(1, -) = 0, \quad P(1, -) = 0, \quad D(1, 0) = \infty, \quad D(1, +) = 1. \]
\[ L(2, -) = 1, \quad P(2, 0) = \infty, \quad P(2, +) = 1, \quad D(2, 0) = \infty, \quad D(2, +) = 2. \]
\[ L(3, -) = 1, \quad P(3, -) = 2, \quad D(3, -) = 2. \]

The values with \( n = 1 \) are easy to justify. \( P(2, +) = 1 \) because any of the coins may be weighed against a reference coin to isolate the defective coin. That \( D(2, +) = 2 \) can be justified by a decision tree argument: there are 4 possible outcomes given two coins where either could be heavy or light, and each weighing has a 3-way outcome; hence, 2 weighings are needed. \( L(3, -) = 1 \) because weighing any two of the coins isolates the defective one. \( P(3, -) = 2 \) can be established by partial enumeration: if the first weighing is among two of the pdc's and they are unequal then at least one more weighing is needed; if the first weighing is between a pdc and a reference coin and they are found to be equal then at least one more weighing is needed to isolate the defective one. \( D(3, -) \geq P(3, -) = 2 \), and a procedure with 2 weighings can be devised for \( D(3, -) \).

\[ n = 4: \quad L(4, -) = 2, \quad P(4, -) = 2, \quad D(4, 0) = 3, \quad D(4, 1) = 2. \]

\( L(4, -) \geq 2 \) from decision tree arguments, and a procedure with 2 weighings can be shown: compare 2 pdc's against the other 2 and apply \( L(2, -) \) to the lighter side. Similar arguments show that \( P(4, -) \geq 2 \), and the following procedure with 2 weighings can be used in this case: compare 1 pdc against another; in case they are unequal apply \( P(2, +) \) to these two pdc's; if they are equal apply \( P(2, +) \) to the remaining two pdc's. To see that \( D(4, 0) = 3 \) we enumerate all possibilities for the first weighing. If one pdc is compared against another and they are found to be equal then \( D(2, +) \), i.e., 2, more weighings are required. If the first comparison is between 2 pdc's in each group – say, between \( a, b \) and \( c, d \) – and \( a, b \) is found to be light then 4 possibilities – i.e., \( a \) light, \( b \) light, \( c \) heavy, \( d \) heavy – have to be distinguished, and 2 more weighings are needed.

The following figure gives a procedure that establishes that \( D(4, 1) \leq 2 \). Here, \( a, b, c, d \) are pdc's and \( x \) is the reference coin. The possibility of \( ab \) being heavier than \( cx \) in the first weighing is not considered because the treatment is analogous to the case where \( ab \) is lighter. Henceforth, a branch is labelled with \( < \) if the left side weighs less than the right side; similarly for the other symbols. Note that if a two sides are equal then all pdc's that being weighed are good, if left side is lighter than the right then either one of the pdc's in the left is light or a pdc in the right side is heavy.

\[ n = 5: \quad L(5, -) = 2, \quad P(5, 0) = 3, \quad P(5, 1) = 2, \quad D(5, -) = 3. \]

The lower bounds are not hard to establish. We show the various procedures for the upper bounds in the following figures. Let the pdc's be named \( a, b, c, d, e \) and the reference coin, if any, be called \( x \). As before, we assume that the treatment for the left pan being heavy is analogous to the case where it is light.
Next, we show that $D(12, 0) = 3$. The lower bound follows using a decision tree argument. The procedure for the upper bound is shown in figure 5 where $a, b, c, d, p, q, r, s$ are pdcs. It can also be shown that $P(13, 0) = 3$: the procedure is as for $D(12, 0)$ except we replace the node $D(4, 1)$ by $P(5, 1)$ which is also 2.
Figure 3: $P(5, 1) \leq 2$

Figure 4: $D(5, -) \leq 3$

Figure 5: $D(12, 0) \leq 3$