Sample Solutions to Homework #0

1. (Section 1.1, Exercise 8(b), page 16) You pass the course if and only if you don’t miss the final exam.

2. (Section 1.1, Exercise 14(b), page 17) Exclusive or. (The restaurant will probably charge more if the diner wants both items.)

3. (Section 1.1, Exercise 18(g), page 17) If you log on to the server, then you have a password.

4. (Section 1.1, Exercise 52, page 20) Define predicate $A$ as: $A = \text{true}$ iff $A$ is a knight. Define predicate $B$ similarly. What $A$ and $B$ said can then be formulated as: $A \leftrightarrow (A \land B)$ and $B \leftrightarrow \neg A$. These two equivalences hold because a knight is always honest (i.e., what a knight says is always true), while a knave always lie (i.e., what a knave says is always false). Hence,

$$A \leftrightarrow (A \land B)$$

$$\equiv A \leftrightarrow (A \land \neg A) \quad \{B \leftrightarrow \neg A\}$$

$$\equiv A \leftrightarrow \text{false}. \quad \{\text{negation law}\}$$

(In the derivation above, \{\} enclose justification for each step.) Therefore, $A$ is a knave, which makes $B$ a knight.

5. (Section 1.2, Exercise 16, page 27)

$$p \rightarrow q$$

$$\equiv \neg p \lor q \quad \{\text{property of } \rightarrow; \text{ Example 3 on page 22}\}$$

$$\equiv q \lor \neg p \quad \{\text{commutativity of } \lor\}$$

$$\equiv \neg(q) \lor \neg p \quad \{\text{double negation law}\}$$

$$\equiv \neg q \rightarrow \neg p \quad \{\text{property of } \rightarrow\}$$

6. (Section 1.2, Exercise 36, page 27) Each line of the truth table corresponds to exactly one combination of truth values for the $n$ atomic propositions involved. Corresponding to each line of the truth table for which the compound proposition is true, we can write a conjunction of the atomic propositions that are true and of the negations of the atomic propositions that are false. We then take the disjunction of these conjunctions and we obtain the desired disjunctive normal form.

7. (Section 1.2, Exercise 38, page 27) Given a compound proposition $p$, by Exercise 37, we can find a proposition $q$ that is logically equivalent to $p$ and uses only $\neg$, $\lor$, and $\land$. Then by De Morgan’s law we can remove all the $\lor$’s by replacing each occurrence of $p_1 \lor p_2 \lor \ldots \lor p_k$ by $\neg(\neg p_1 \land \neg p_2 \land \ldots \land \neg p_k)$. 
8. (Section 1.3, Exercise 23(f), page 41) Let $F(x)$ be “$x$ is your friend” and let $P(x)$ be “$x$ is perfect.” Then the statement can be expressed as $\neg(\forall x F(x)) \lor (\exists x \neg P(x))$. (Note that the answer in the textbook is not quite correct.)

9. (Section 1.3, Exercise 46, page 43) It suffices to find a counterexample. Let the universe of $x$ be integers, let $P(x)$ be “$x$ is even”, and let $Q(x)$ be “$x$ is odd.” Then $(\forall x P(x)) \lor (\exists x Q(x))$ is false, while $(\exists x \neg P(x))$ is true.

10. (Section 1.4, Exercise 14(e), page 53) Let $x$ and $y$ be integers, let $z = x + y$. Then the statement can be expressed as $x + y = z$. Then $(\exists x \exists y (O(y, z) \rightarrow T(x, y)))$.

11. (Section 1.4, Exercise 50, page 56) The statement is $\forall \exists \forall \forall (n > N \rightarrow |a_n - L| < \varepsilon)$, where $\varepsilon$ ranges over positive reals, and $n$ and $N$ range over positive integers. (See also Example 10 on page 47.)

12. (Section 1.5, Exercise 20, page 75) The assertion to be proved is that for all integers $n$, if $n$ is even, then $n^2$ is even. We will use the fact that if $n$ is even iff $n = 2k$ for some $k$.

- (a) Direct proof: If $n$ is even, then $n = 2k$ for some $k$. Hence, $n^2 = (2k)^2 = 2(2k^2)$. Thus $n^2$ is even.
- (b) Indirect proof: If $n^2$ is not even (i.e., $n^2$ is odd), by Example 19 on page 65, we conclude that $n$ is odd.
- (c) Proof by contradiction: Suppose $n^2$ is not even (i.e., $n^2$ is odd). By Example 19 on page 65, this assumption implies $n$ is odd, which contradicts our other assumption that $n$ is even. Thus, $n^2$ is even.

13. (Section 1.5, Exercise 58, page 76) Given $x$, let $n$ be the greatest integer at most $x$ (i.e., $n = \lfloor x \rfloor$), and let $\varepsilon = x - n$. Clearly $0 \leq \varepsilon < 1$, and $\varepsilon$ is unique for this $n$. Any other choice of $n$ will cause $\varepsilon < 0$ or $\varepsilon \geq 1$. Hence $n$ is also unique.

14. (Section 1.5, Exercise 72, page 76) Denote the ten integers on the circle, clockwise from an arbitrary position, by $a_0, a_1, \ldots, a_9$. Let $S_i = a_i + a_{(i+1) \mod 10} + a_{(i+2) \mod 10}$, for $0 \leq i \leq 9$ (i.e., $S_i$ is the sum of the three integers starting from $a_i$). Hence,

$$\sum_{i=0}^{9} S_i = 3 \sum_{i=0}^{9} a_i = 3 \cdot 55 = 165.$$ 

Therefore, the average of $S_i$, where $0 \leq i \leq 9$, is 16.5. By the averaging argument, there exists a $k$ such that $S_k \geq 16.5$. Since $S_k$ is an integer, we conclude $S_k \geq 17$.

15. (Section 1.6, Exercise 8(a), page 85) True.

16. (Section 1.6, Exercise 14(d), page 85) 3.

17. (Section 1.6, Exercise 22, page 85) We conclude $A = \emptyset$ or $B = \emptyset$. We prove this by contradiction. Suppose neither $A$ nor $B$ is empty. Then $A$ has at least one element $a$ and $B$ has at least one element $b$. Thus $A \times B$ has at least one element $(a, b)$, contradicting that $A \times B = \emptyset$. Therefore, $A = \emptyset$ or $B = \emptyset$.

18. (Section 1.6, Exercise 30, page 86)
(a) If \( S \in S \), then by the defining condition for \( S \), we conclude that \( S \not\in S \), a contradiction.

(b) If \( S \not\in S \), then by the defining condition for \( S \), we conclude that \( S \not\in S \), again a contradiction.

19. (Section 1.7, Exercise 32, page 95) Yes. Each side consists of those elements that are in odd number of the sets \( A \), \( B \), and \( C \).

20. (Section 1.7, Exercise 36, page 95) We count \(|A \cup B \cup C|\) as follows. First we count the elements in each set and add. But we have overcounted: each element in \( A \cap B \), \( A \cap C \), and \( B \cap C \) has been counted twice. Therefore, we subtract the cardinalities of these intersections to make up for the overcount. However, we then have undercounted: the elements of \( A \cap B \cap C \) has been added three times and subtracted three times. Therefore, we add back \(|A \cap B \cap C|\).

An alternative solution is as follows.

\[
|A \cup B \cup C| = \{\text{consider } B \cup C \text{ as a set; principle of inclusion-exclusion for two sets, page 87}\} \\
|A| + |B \cup C| - |A \cap (B \cup C)| \\
= \{\text{principle of inclusion-exclusion for two sets}\} \\
|A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| \\
= \{\text{distributive law, page 89}\} \\
|A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |A \cap B \cap C|) \\
= \{\text{arithmetic}\} \\
|A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\
\]

21. (Section 1.7, Exercise 38, page 95) (a) \( A_n = \{\ldots, -1, 0, 1, \ldots, n\} \) (b) \( A_1 = \{\ldots, -1, 0, 1\} \).

22. (Section 1.8, Exercise 18(c), page 109) This function is a bijection. But not from \( \mathbb{R} \) to \( \mathbb{R} \), because \( x = 2 \) is not in the domain and \( f(x) = 1 \) is not in the range. Function \( f \) is a bijection from \( \mathbb{R} - \{-2\} \) to \( \mathbb{R} - \{1\} \).

23. (Section 1.8, Exercise 26, page 109) Suppose that \( g : A \to B \) and \( f : B \to C \), so \( f \circ g : A \to C \). We will prove that \( g \) is one-to-one by contradiction. Suppose \( g \) were not one-to-one. By definition, this implies that there exist distinct elements \( a_1 \) and \( a_2 \) such that \( g(a_1) = g(a_2) \). Then \( f(g(a_1)) = f(g(a_2)) \), that is, \( (f \circ g)(a_1) = (f \circ g)(a_2) \), which implies \( f \circ g \) is not one-to-one. A contradiction. Therefore, \( g \) is one-to-one.

24. (Section 1.8, Exercise 48, page 110) The largest integer at most \( b \) is \( \lfloor b \rfloor \); the smallest integer at least \( a \) is \( \lceil a \rceil \). Therefore, the number of integers between \( a \) and \( b \), inclusive, is \( \lfloor b \rfloor - \lceil a \rceil + 1 \).

25. (Section 1.8, Exercise 69(a), page 111) Domain is \( \mathbb{Z} \); codomain is \( \mathbb{R} \); domain of definition is the set of nonzero integers; the set of values for which \( f \) is undefined is \( \{0\} \); not a total function.