Test #2

Instructions. This is a 75-minute test. There are seven questions worth a total of 60 points. All program variables appearing in this test should be assumed to be of type integer.

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1. **(8 points total)** In each of the following two parts, let $S$ be a given program and let $q$ be a given postcondition.

   (a) (4 points) Characterize the set of program states belonging to $wp(S, q)$ in terms of execution of program $S$. (Remark: Recall that $wp(S, q)$ is a state predicate, and as such it may be viewed as a set of program states.)

   (b) (4 points) What are the two main steps involved in establishing the partial correctness of $S$ with respect to precondition $p$ and postcondition $q$ using the weakest precondition method?

2. **(8 points total)**

   (a) (4 points) Explain how to determine the weakest precondition of a program $S$ with respect to a given postcondition $q$ in the case where $S$ consists of a single assignment statement.

   (b) (4 points) Use the approach of part (a) to compute $wp(x := x + y, x > 2y)$. Simplify your final answer as much as possible.
3. (12 points total)

(a) (4 points) Let $S_1$ and $S_2$ be two programs and let $S = S_1; S_2$. Let $q$ be a given postcondition. Express $wp(S, q)$ as a function of $S_1$, $S_2$, and $q$ (and using one or more applications of $wp$).

(b) (4 points) Let $S_1$, $S_2$, and $S_3$ be three programs and let $S = S_1; S_2; S_3$. Let $q$ be a given postcondition. Express $wp(S, q)$ as a function of $S_1$, $S_2$, $S_3$, and $q$ (and using one or more applications of $wp$).

(c) (4 points) Use the approach of part (b) to compute $wp(S, x > y)$ where $S$ is $z := x + y; x := y + z; y := x + z$. Simplify your final answer as much as possible.

4. (8 points total)

(a) (4 points) Let $S_1$ and $S_2$ be two programs, let $C$ be a boolean condition, and let $S$ be the program $\text{if } (C) \text{ then } S_1 \text{ else } S_2$. Let $q$ be a given postcondition. Express $wp(S, q)$ as a function of $S_1$, $S_2$, and $q$ (and using one or more applications of $wp$).

(b) (4 points) Use the approach of part (a) to compute $wp(S, x > y)$ where $S$ is the program $\text{if } (x \leq y) \text{ then } x := y \text{ else } x := x + y$. Simplify your final answer as much as possible.
5. (10 points) Let \( S_1 \) and \( S_2 \) be two programs, let \( C \) be a boolean condition, and let \( S \) be the program \( S_1 \); \textbf{while} \( (C) \) \( S_2 \). Suppose that we wish to use the weakest preconditon method along with the loop invariant technique to prove the partial correctness of \( S \) with respect to a given precondition \( p \) and postcondition \( q \). What are the main steps of such a proof?

6. (4 points) Give an example of a program \( S \), precondition \( p \), and postcondition \( q \) such that \( S \) is partially correct, but not totally correct, with respect to precondition \( p \) and postcondition \( q \).
7. (10 points) Use a potential function argument to prove that the following program terminates from any initial program state.

\[ m := 0; \]
\[ \textbf{while} \ (n > 0) \]
\[ \quad \textbf{if} \ (m < n) \ \textbf{then} \]
\[ \quad \quad m := m + 1 \]
\[ \quad \textbf{else} \{ \]
\[ \quad \quad m := 0; \]
\[ \quad \quad n := n - 1 \]
\[ \} \]

It can be shown that the state predicate

\[ 0 \leq m \leq n \]

is a loop invariant of the above \texttt{while} loop; feel free to make use of this fact in your proof. Hint: Make use of the potential function

\[ f(m, n) = \frac{(n + 1)(n + 2)}{2} - m. \]