Vector Spaces
Examples of vector spaces:

- $\mathbb{R}$:
- $\mathbb{R}^2$:
- $\mathbb{R}^3$:
- $\mathbb{R}^n$: Set of vectors of the form $x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$, where $\chi_0, \chi_1, \ldots, \chi_{n-1} \in \mathbb{R}$.

  - Direction (vector) from the origin (the point $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$) to the point $x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$.

  - A direction is position independent: You can think of it as a direction anchored anywhere in $\mathbb{R}^n$.

http://z.cs.utexas.edu/wiki/pla.wiki/
Definition

- Let $S$ be a set.
- This set is called a space if
  - There is a notion of multiplying an element in the set by a scalar: if $x \in S$ and $\alpha \in \mathbb{R}$ then $\alpha x$ is defined.
  - Scaling an element in the set always results in an element of that set: if $x \in S$ and $\alpha \in \mathbb{R}$ then $\alpha x \in S$.
  - There is a notion of adding elements in the set by a scalar: if $x, y \in S$ then $x + y$ is defined.
  - Adding two elements in the set always results in an element of that set: if $x, y \in S$ then $x + y \in S$.

Note

A space has to have the notion of a zero element.
Example

Let $x, y \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

- $z = x + y \in \mathbb{R}^2$;
- $\alpha \cdot x = \alpha x \in \mathbb{R}^2$; and
- $0 \in \mathbb{R}^2$ and $0 \cdot x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Vector spaces

We will talk about vector spaces because the spaces have vectors as their elements.
Example

- Consider the set of all real valued $m \times n$ matrices, $\mathbb{R}^{m \times n}$.
- Together with matrix addition and multiplication by a scalar, this set is a vector space.
- Note that an easy way to visualize this is to take the matrix and view it as a vector of length $m \cdot n$. 
Example

- Not all spaces are vector spaces.
- The spaces of all functions defined from $\mathbb{R}$ to $\mathbb{R}$ has addition and multiplication by a scalar defined on it, but it is not a vectors space.
- It is a space of functions instead.
Subspaces

- Recall the concept of a subset, \( B \), of a given set, \( A \).
- All elements in \( B \) are elements in \( A \).
- If \( A \) is a vector space we can ask ourselves the question of when \( B \) is also a vector space.

Answer

The answer is that \( B \) is a vector space if

- \( x, y \in B \) implies that \( x + y \in B \)
- \( x \in B \) and \( \alpha \in B \) implies \( \alpha x \in B \)
- \( 0 \in B \) (the zero vector).

We call a subset of a vector space that is also a vector space a subspace.
Definition

Let $A$ be a vector space and let $B$ be a subset of $A$. Then $B$ is a subspace of $A$ if

- $x, y \in B$ implies that $x + y \in B$.
- $x \in B$ and $\alpha \in \mathbb{R}$ implies that $\alpha x \in B$. 

http://z.cs.utexas.edu/wiki/pla.wiki/
Note
One way to describe a subspace is that it is a subset that is closed under addition and scalar multiplication.

Example
The empty set is a subset of \( \mathbb{R}^n \). Is it a subspace? Why?
Exercise

What is the smallest subspace of $\mathbb{R}^n$? (Smallest in terms of the number of elements in the subspace.)
Why Should We Care?
Example

Consider

\[ A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix} \]

Does \( Ax = b_0 \) have a solution?
Example

Consider

\[
A = \begin{pmatrix}
3 & -1 & 2 \\
1 & 2 & 0 \\
4 & 1 & 2
\end{pmatrix}, \quad b_0 = \begin{pmatrix}
8 \\
-1 \\
7
\end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix}
5 \\
-1 \\
7
\end{pmatrix}
\]

- Does \( Ax = b_0 \) have a solution?

- The answer is yes: \( x = \begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix} \).
Example

Consider

\[ A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix} \]

Does \( Ax = b_0 \) have a solution?

The answer is yes: \( x = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{3}{2} \end{pmatrix} \).

Does \( Ax = b_1 \) have a solution?

http://z.cs.utexas.edu/wiki/pla.wiki/
Example

Consider

\[
A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}
\]

- Does \(Ax = b_0\) have a solution?
- The answer is yes: \(x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\).
- Does \(Ax = b_1\) have a solution? The answer is no.
Example

Consider

\[ A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix} \]

- Does \( Ax = b_0 \) have a solution?
  - The answer is yes: \( x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \).

- Does \( Ax = b_1 \) have a solution? The answer is no.
- Does \( Ax = b_0 \) have any other solutions?
Example

Consider

\[ A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix} \]

- Does \( Ax = b_0 \) have a solution?
  - The answer is yes: \( x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \).

- Does \( Ax = b_1 \) have a solution? The answer is no.

- Does \( Ax = b_0 \) have any other solutions? The answer is yes.
Example

Consider

\[ A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix} \]

- Does \( Ax = b_0 \) have a solution?
  - The answer is yes: \( x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \).
- Does \( Ax = b_1 \) have a solution? The answer is no.
- Does \( Ax = b_0 \) have any other solutions? The answer is yes.
- How do we characterize all solutions?
Example

Consider the points \((1, 3), (2, -2), (4, 1)\).

- Is there a straight line through these points?
- Under what circumstances is there a straight line through these points?
Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $Ax = b$. Partition

$$A \rightarrow (a_0 | a_1 | \cdots | a_{n-1}) \quad \text{and} \quad x \rightarrow \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}.$$ 

Then

$$\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1} = b.$$ 

**Definition**

Let $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^m$ and $\{\chi_0, \ldots, \chi_{n-1}\} \subset \mathbb{R}$. Then

$$\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}$$

is said to be a *linear combination* of the vectors $\{a_0, \ldots, a_{n-1}\}$.
Consider $Ax = b$.

Solution $x$ exists if and only if $b$ equals a linear combination of the columns of $A$.

This observation answers the question “Given a matrix $A$, for what right-hand side vector, $b$, does $Ax = b$ have a solution?”

The answer is that there is a solution if and only if $b$ is a linear combination of the columns (column vectors) of $A$. 

http://z.cs.utexas.edu/wiki/pla/wiki/
Definition

The column space of $A \in \mathbb{R}^{m \times n}$ is the set of all vectors $b \in \mathbb{R}^m$ for which there exists a vector $x \in \mathbb{R}^n$ such that $Ax = b$. 
The column space of $A \in \mathbb{R}^{m \times n}$ is a subspace (of $\mathbb{R}^m$).

Proof

- The column space of $A$ is closed under addition:
  - Let $b_0, b_1 \in \mathbb{R}^m$ be in the column space of $A$.
  - Then there exist $x_0, x_1 \in \mathbb{R}^n$ such that $Ax_0 = b_0$ and $Ax_1 = b_1$.
  - But then $A(x_0 + x_1) = Ax_0 + Ax_1 = b_0 + b_1$ and thus $b_0 + b_1$ is in the column space of $A$.

- The column space of $A$ is closed under scalar multiplication:
  - Let $b$ be in the column space of $A$ and $\alpha \in \mathbb{R}$.
  - Then there exists a vector $x$ such that $Ax = b$.
  - Hence $\alpha Ax = \alpha b$.
  - Since $A(\alpha x) = \alpha Ax = \alpha b$ we conclude that $\alpha b$ is in the column space of $A$.

Hence the column space of $A$ is a subspace (of $\mathbb{R}^m$).
Example

\[ A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}. \]

- Set two appended systems:

\[
\begin{pmatrix} 3 & -1 & 2 & 8 \\ 1 & 2 & 0 & -1 \\ 4 & 1 & 2 & 7 \end{pmatrix} \quad \begin{pmatrix} 3 & -1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 1 & 2 & 0 \end{pmatrix}. \quad (1)
\]

- It becomes convenient to swap the first and second equation:

\[
\begin{pmatrix} 1 & 2 & 0 & -1 \\ 3 & -1 & 2 & 8 \\ 4 & 1 & 2 & 7 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & -1 & 2 & 0 \\ 4 & 1 & 2 & 0 \end{pmatrix}.
\]
Continued

\[
\begin{pmatrix}
1 & 2 & 0 & -1 \\
3 & -1 & 2 & 8 \\
4 & 1 & 2 & 7
\end{pmatrix}
\quad \begin{pmatrix}
1 & 2 & 0 & 0 \\
3 & -1 & 2 & 0 \\
4 & 1 & 2 & 0
\end{pmatrix}
\quad .
\]

\[
\begin{pmatrix}
1 & 2 & 0 & -1 \\
0 & -7 & 2 & 11 \\
0 & -7 & 2 & 11
\end{pmatrix}
\quad \begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & -7 & 2 & 0 \\
0 & -7 & 2 & 0
\end{pmatrix}
\quad .
\]

\[
\begin{pmatrix}
1 & 2 & 0 & -1 \\
0 & -7 & 2 & 11 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & -7 & 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad .
\]

\[
\begin{pmatrix}
1 & 2 & 0 & -1 \\
0 & 1 & -2/7 & -11/7 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 1 & -2/7 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad .
\]

\[
\begin{pmatrix}
1 & 0 & 4/7 & 15/7 \\
0 & 1 & -2/7 & -11/7 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 4/7 & 0 \\
0 & 1 & -2/7 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad .
\]
\[
\begin{pmatrix}
1 & 0 & 4/7 & 15/7 \\
0 & 1 & -2/7 & -11/7 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 4/7 & 0 \\
0 & 1 & -2/7 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

- \(0 \times \chi_2 = 0\): no constraint on variable \(\chi_2\).
- \(\chi_2\) is a free variable.
- \(\chi_1 - 2/7\chi_2 = -11/7\), or,

\[
\chi_1 = -11/7 + 2/7\chi_2
\]

The value of \(\chi_1\) is constrained by the value given to \(\chi_2\).
- \(\chi_0 + 4/7\chi_2 = 15/7\), or,

\[
\chi_0 = 15/7 - 4/7\chi_2.
\]

Thus, the value of \(\chi_0\) is constrained by the value given to \(\chi_2\).
Example (continued)

\[
\begin{pmatrix}
1 & 0 & 4/7 & 15/7 \\
0 & 1 & -2/7 & -11/7 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 4/7 & 0 \\
0 & 1 & -2/7 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

\[
\begin{pmatrix}
15/7 - 4/7\chi_2 \\
-11/7 + 2/7\chi_2 \\
\chi_2
\end{pmatrix}
\]
solves the linear system.

We can rewrite this as

\[
\begin{pmatrix}
15/7 \\
-11/7 \\
0
\end{pmatrix}
+ \chi_2
\begin{pmatrix}
-4/7 \\
2/7 \\
1
\end{pmatrix}.
\]

So, for each choice of \(\chi_2\), we get a solution to the linear system.
Note

\[
\begin{pmatrix}
3 & -1 & 2 \\
1 & 2 & 0 \\
4 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
\frac{15}{7} \\
-\frac{11}{7} \\
0
\end{pmatrix}
= \begin{pmatrix}
8 \\
-1 \\
7
\end{pmatrix}.
\]

- \( x_p \) is a \textit{particular} solution to \( Ax = b_0 \). (Hence the \( p \) in the \( x_p \).)

- Note that

\[
\begin{pmatrix}
3 & -1 & 2 \\
1 & 2 & 0 \\
4 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
-\frac{4}{7} \\
\frac{2}{7} \\
1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

- For \textit{any} \( \alpha \), \( (x_p + \alpha x_n) \) is a solution to \( Ax = b_0 \):

\[
A(x_p + \alpha x_n) = Ax_p + A(\alpha x_n) = Ax_p + \alpha Ax_n = b_0 + \alpha \times 0 = b_0.
\]
Example (continued)

- The system $Ax = b_0$ has **an infinite number of** solutions.
- To characterize all solutions, it suffices to find one (nonunique) particular solution $x_p$ that satisfies $Ax_p = b_0$.
- Now, for any vector $x_n$ that has the property that $Ax_n = 0$, we know that $x_p + x_n$ is also a solution.

Definition

Let $A \in \mathbb{R}^{m \times n}$. Then the set of all vectors $x \in \mathbb{R}^n$ that have the property that $Ax = 0$ is called the **null space** of $A$ and is denoted by $\mathcal{N}(A)$. 
**Theorem**

The null space of $A \in \mathbb{R}^{m \times n}$ is indeed a subspace of $\mathbb{R}^n$.

**Proof**

- Clearly $\mathcal{N}(A)$ is a subset of $\mathbb{R}^n$.
- Show that it is closed under addition:
  - Assume that $x, y \in \mathcal{N}(A)$.
  - Then $A(x + y) = Ax + Ay = 0$
  - Therefore $(x + y) \in \mathcal{N}(A)$.
- Show that it is closed under addition:
  - Assume that $x \in \mathcal{N}(A)$ and $\alpha \in \mathbb{R}$.
  - Then $A(\alpha x) = \alpha Ax = \alpha \times 0 = 0$
  - Therefore $\alpha x \in \mathcal{N}(A)$.
- Hence, $\mathcal{N}(A)$ is a subspace.

The zero vector (of appropriate length) is always in the null space of a matrix $A$. 

http://z.cs.utexas.edu/wiki/pla.wiki/
Systematic Steps to Finding All Solutions
Example

Consider

\[
\begin{pmatrix}
1 & 3 & 1 & 2 \\
2 & 6 & 4 & 8 \\
0 & 0 & 2 & 4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
t
\end{pmatrix}
= \begin{pmatrix}
1 \\
3 \\
1
\end{pmatrix}
\]
Reduction to Row-Echelon Form

\[
\begin{pmatrix}
1 & 3 & 1 & 2 & 1 \\
2 & 6 & 4 & 8 & 3 \\
0 & 0 & 2 & 4 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Identify the pivots

The pivots are the first nonzero elements in each row to the left of the line:

\[
\begin{pmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Give the general solution to the problem

\[
\begin{pmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

- Identify the free variables (the variables corresponding to the columns that do not have pivots in them):
  - \(y\) and \(t\)

- A general solution can be given by

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} + \alpha \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} + \beta \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]
Compute \( x_p \)

\[
\begin{pmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & \box{2} & 4 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

\( x_p \) is a particular (special) solution.

Set the free variables to zero and solve:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \box{1} \\
0 & 0 & 2 & 0 & \box{2} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
0 \\
z \\
0
\end{pmatrix}
= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix} x + z \\ 2z \end{pmatrix} = 1
\]

Solving this yields \( x_p = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} \).
Compute $x_{n_0}$

$$
\begin{pmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

$\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}$ is the form of one vector in the null space.

Solve

$$
\begin{pmatrix}
1 & 3 & 1 & 2 \\
0 & 0 & 2 & 4
\end{pmatrix}\begin{pmatrix} x \\ 1 \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ or } \cdots
$$

Solving this yields $x_{n_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Compute $x_{n_1}$

\[
\begin{pmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & 2 & 4 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

$x_{n_1} = \begin{pmatrix} \square \\ 0 \\ \square \\ 1 \end{pmatrix}$ is the form of another vector in the null space.

Solve

\[
\begin{pmatrix}
1 & 3 & 1 & 2 \\
0 & 0 & 2 & 4 \\
\end{pmatrix} \begin{pmatrix}
x \\
0 \\
z \\
1 \\
\end{pmatrix} = \begin{pmatrix}
1 \\
1 \\
0 \\
\end{pmatrix} \text{ or } \cdots
\]

Solving this yields $x_{n_0} = \begin{pmatrix} \square \\ 0 \\ \square \\ 1 \end{pmatrix}$.