CS311: Discrete Math for Computer Science, Spring 2015

Homework Assignment 4, with Solutions

1. Find coefficients \(a, b, c, d\) such that formula (5) from Part 3 of Lecture Notes is satisfied for all values of \(n\).

   Solution: By substituting 0, 1, 2 and 3 for \(n\) in the formula
   
   \[S_n = an^3 + bn^2 + cn + d,\]
   
   we get the equations
   
   \[
   \begin{align*}
   0 &= d, \\
   1 &= a + b + c + d, \\
   5 &= 8a + 4b + 2c + d, \\
   14 &= 27a + 9b + 3c + d.
   \end{align*}
   \]

   From these equations we find: \(a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}, d = 0\).

2. (a) Find a formula for
   
   \[
   \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}
   \]
   
   by examining the values of this expression for small values of \(n\).

   Answer: \(\frac{n}{n+1}\).

   (b) Prove the formula you conjectured in part (a).

   Solution: we will prove the formula
   
   \[
   \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}
   \]
   
   by induction. Basis: When \(n = 0\), the formula turns into 0 = \(\frac{0}{1}\). Induction step: assuming that the given formula holds for \(n\), we can prove that
   
   \[
   \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n+1}{n+2}
   \]
   
   as follows:
   
   \[
   \begin{align*}
   \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1) \cdot (n+2)} \\
   &= \frac{n \cdot (n+2) + 1}{(n+1) \cdot (n+2)} = \frac{n^2 + 2n + 1}{(n+1) \cdot (n+2)} = \frac{(n+1)^2}{(n+1) \cdot (n+2)} = \frac{n + 1}{n + 2}.
   \end{align*}
   \]
3. (a) Find a formula for
\[ 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! \]
by examining the values of this expression for small values of \( n \).

Answer: \((n + 1)! - 1\).

(b) Prove the formula you conjectured in part (a).

Solution: we will prove the formula
\[ 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1 \]
by induction. Basis: When \( n = 0 \), the formula turns into \( 0 = 1! - 1 \). Induction step: assuming that the given formula holds for \( n \), we can prove that
\[ 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n + 1) \cdot (n + 1)! = (n + 2)! - 1 \]
as follows:
\[
1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n + 1) \cdot (n + 1)! = (n + 1)! - 1 + (n + 1) \cdot (n + 1)!
= (n + 1)!(n + 1 + 1) - 1
= (n + 2)! - 1.
\]

4. Prove that for every nonnegative integer \( n \)
\[ \sum_{i=1}^{n} i2^i = (n - 1)2^{n+1} + 2. \]

Solution: we will prove the formula by induction. Basis: When \( n = 0 \), the formula turns into \( 0 = (-1) \cdot 2 + 2 \). Induction step: assuming that the given formula holds for \( n \), we can prove that
\[ \sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2 \]
as follows:
\[
\sum_{i=1}^{n+1} i2^i = \sum_{i=1}^{n} i2^i + (n + 1)2^{n+1} = (n - 1)2^{n+1} + 2 + (n + 1)2^{n+1} = 2n \cdot 2^{n+1} + 2 = n2^{n+2} + 2.
\]