Part 3. Triangular Numbers and Their Relatives

Definitions

In the definitions below, $n$ is a nonnegative integer.

The **triangular number** $T_n$ is the sum of all integers from 1 to $n$:

$$T_n = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n.$$  

For instance, $T_4 = 1 + 2 + 3 + 4 = 10$.

By $B_n$ we denote the number of ways to choose two elements out of $n$. For instance, if we take 5 elements $a, b, c, d, e$, then there will be 10 ways to choose two:

$$a, b; a, c; a, d; a, e; b, c; b, d; b, e; c, d; c, e; d, e.$$  

We conclude that $B_5 = 10$.

By $P_n$ we denote the number of parts into which a plane is divided by $n$ straight lines in general position. (“In general position” means that there are no parallel lines and no multiple intersection points.) For instance, 3 lines in general position divide the plane into 7 parts: a triangle and 6 infinite regions. We conclude that $P_3 = 7$.

By $S_n$ we denote the sum of the squares of all integers from 1 to $n$:

$$S_n = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2.$$  

For instance, $S_4 = 1^2 + 2^2 + 3^2 + 4^2 = 30$.

By $C_n$ we denote the sum of the cubes of all integers from 1 to $n$:

$$C_n = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3.$$  

For instance, $C_4 = 1^3 + 2^3 + 3^3 + 4^3 = 100$.

The **harmonic number** $H_n$ is defined by the formula

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$  

For instance, \( H_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}. \)

The factorial of \( n \) is the product of all integers from 1 to \( n \):

\[
n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdots n.
\]

For instance, \( 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \). Note that \( 0! = 1 \).

**A Formula for Triangular Numbers**

Triangular numbers can be calculated using the formula

\[
T_n = \frac{n(n + 1)}{2}. 
\] (1)

To prove this formula, we start with two expressions for \( T_n \):

\[
T_n = 1 + 2 + 3 + \cdots + (n - 2) + (n - 1) + n
\]

\[
T_n = n + (n - 1) + (n - 2) + \cdots + 3 + 2 + 1.
\]

If we add them column by column, we’ll get:

\[
2T_n = (n + 1) + (n + 1) + (n + 1) + \cdots + (n+1) + (n+1) + (n+1) = n(n+1),
\]

and it remains to divide both sides by 2.

There are also other ways to prove formula (1). Consider these identities:

\[
(1 + 1)^2 = 1^2 + 2 \cdot 1 \cdot 1 + 1^2,
\]

\[
(2 + 1)^2 = 2^2 + 2 \cdot 2 \cdot 1 + 1^2,
\]

\[
(3 + 1)^2 = 3^2 + 2 \cdot 3 \cdot 1 + 1^2,
\]

\[
\cdots
\]

\[
(n + 1)^2 = n^2 + 2 \cdot n \cdot 1 + 1^2.
\]

If we add them column by column, we’ll get:

\[
2^2 + 3^2 + 4^2 + \cdots + (n+1)^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 + 2T_n + n.
\]

Subtract \( 2^2 + 3^2 + \cdots + n^2 \) from both sides:

\[
(n + 1)^2 = 1^2 + 2T_n + n.
\]

Expand the left-hand side and subtract \( n + 1 \) from both sides:

\[
n^2 + n = 2T_n.
\]

It remains to divide both sides by 2.
A Formula for $B_n$

To find out how the sequence $B_n$ is related to triangular numbers, we made a table that shows several members of this sequence and several triangular numbers next to each other:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$B_n$</th>
<th>$T_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

We see that for $n = 1, \ldots, 5$,

$$B_n = T_{n-1}. \tag{2}$$

Formula (2) holds actually for all positive integers $n$. This fact can be proved as follows. Number $B_n$ can be thought of as the number of 2-element sets formed from the numbers $1, 2, \ldots, n$. Among these sets, $n - 1$ include the number $n$, because there are $n - 1$ choices for the second element of the set—it can be any of the numbers $1, 2, \ldots, n - 1$. In addition, there are $n - 2$ sets that don’t include $n$ but include $n - 1$, because there are $n - 2$ choices for the second element of the set—it can be any of the numbers $1, 2, \ldots, n - 2$. Similarly, there are $n - 3$ sets that include neither $n$ nor $n - 1$ but include $n - 2$, and so on. Finally, there is one set that doesn’t include any of the numbers greater than 2; that set consists of 1 and 2. It follows that the total number of 2-element sets formed from the numbers $1, 2, \ldots, n$ is

$$(n - 1) + (n - 2) + (n - 3) + \cdots + 1,$$

which is the triangular number $T_{n-1}$.

From formulas (1) and (2) we conclude that

$$B_n = \frac{n(n - 1)}{2}.$$

A Formula for $P_n$

To find out how the sequence $P_n$ is related to triangular numbers, we made a table that shows several members of this sequence and several triangular numbers next to each other:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$P_n$</th>
<th>$T_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>10</td>
</tr>
</tbody>
</table>
We see that for \( n = 0, \ldots, 4 \),
\[
P_n = T_n + 1. \tag{3}
\]

We will prove later that formula (3) holds actually for all \( n \). For the time being, this claim remains a conjecture.

Using formula (1), we can derive from this conjecture that
\[
P_n = \frac{n(n+1)}{2} + 1 = \frac{n^2}{2} + \frac{n}{2} + 1.
\]

A Formula for \( C_n \)
To find out how the sequence \( C_n \) is related to triangular numbers, we made a table that shows several members of this sequence and several triangular numbers next to each other:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( C_n )</th>
<th>( T_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>10</td>
</tr>
</tbody>
</table>

We see that for \( n = 0, \ldots, 4 \),
\[
C_n = (T_n)^2. \tag{4}
\]

We will prove later that formula (4) holds actually for all \( n \). For the time being, this claim remains a conjecture.

Using formula (1), we can derive from this conjecture that
\[
C_n = \frac{n^2(n+1)^2}{4}.
\]

Towards a Formula for \( S_n \)
Formulas (1) and (4) show that \( T_n \) can be expressed as a quadratic polynomial, and that \( C_n \) can be expressed as a polynomial of degree 4:
\[
T_n = \frac{1}{2}n^2 + \frac{1}{2}n,
\]
\[
C_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2.
\]

This observation leads to the conjecture that \( S_n \) can be expressed as a polynomial of degree 3:
\[
S_n = an^3 + bn^2 + cn + d. \tag{5}
\]

We will see later that such a formula indeed exists. The values of \( a, b, c, d \) can be found by the method of method of undetermined coefficients.
How Large are Harmonic Numbers and Factorials?

There are no simple precise formulas for harmonic numbers and factorials. About the sequence $H_n$ we know that it goes to infinity, but very slowly. There is a simple approximate formula for $H_n$, and we will discuss it later in this course.

It is interesting to compare factorials with the powers of 2:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n!$</th>
<th>$2^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>32</td>
</tr>
</tbody>
</table>

We see that the factorial of $n$ is greater than $2^n$ when $n$ is 4 or 5, but it is less than or equal to $2^n$ for smaller values of $n$. We will prove later that $n! > 2^n$ for all values of $n$ beginning with 4.