Part 7. Sets, Relations and Functions

Sets

A set is a collection of objects. We write $x \in A$ if object $x$ is an element of set $A$, and $x \notin A$ otherwise.

The set whose elements are $x_1, \ldots, x_n$ is denoted by $\{x_1, \ldots, x_n\}$. The set $\{\}$ is called empty and denoted also by $\emptyset$. The set of nonnegative integers is denoted by $\mathbb{N}$:

$$\mathbb{N} = \{0, 1, 2, \ldots\}.$$  

Other examples are the set $\mathbb{Z}$ of all integers and the set $\mathbb{R}$ of real numbers.

When we specify which objects belong to a set, this defines the set completely; there is no such thing as the order of elements in a set or the number of repetitions of an element in a set. For instance,

$$\{2, 3\} = \{3, 2\} = \{2, 2, 3\}.$$  

If $C$ is a condition, then by $\{x \mid C\}$ we denote the set of all objects $x$ satisfying this condition. For instance,

$$\{x \mid x = 2 \lor x = 3\}$$

is the same set as $\{2, 3\}$.

If $A$ is a set and $C$ is a condition, then by $\{x \mid x \in A ; C\}$ we denote the set of all elements of $A$ satisfying condition $C$. For instance, $\{2, 3\}$ can be also written as

$$\{x \in \mathbb{N} \mid 1 < x < 4\}.$$  

If a set $A$ is finite then the number of elements of $A$ is also called the cardinality of $A$ and denoted by $|A|$. For instance,

$$|\emptyset| = 0, \quad |\{2, 3\}| = 2.$$  

We say that a set $A$ is a subset of a set $B$, and write $A \subseteq B$, if every element of $A$ is an element of $B$. For instance,

$$\emptyset \subseteq \mathbb{N}, \quad \{2,3\} \subseteq \mathbb{N}.$$
Operations on Sets

For any sets $A$ and $B$, by $A \cup B$ we denote the set
\[
\{ x : x \in A \lor x \in B \},
\]
called the union of $A$ and $B$. By $A \cap B$ we denote the set
\[
\{ x : x \in A \land x \in B \},
\]
called the intersection of $A$ and $B$. For instance,
\[
\{2, 3\} \cup \{3, 5\} = \{2, 3, 5\},
\]
\[
\{2, 3\} \cap \{3, 5\} = \{3\}.
\]

By $A \setminus B$ we denote the set
\[
\{ x : x \in A \land x \notin B \},
\]
called the difference of $A$ and $B$. For instance,
\[
\{2, 3\} \setminus \{3, 5\} = \{2\}.
\]

The Cartesian product of sets $A$ and $B$ is the set of ordered pairs $\langle x, y \rangle$ such that $x \in A$ and $y \in B$:
\[
A \times B = \{ \langle x, y \rangle : x \in A \land y \in B \}.
\]
For instance,
\[
\{1, 2\} \times \{2, 3, 4, 5, 6\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 1, 4 \rangle, \langle 1, 5 \rangle, \langle 1, 6 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 2, 4 \rangle, \langle 2, 5 \rangle, \langle 2, 6 \rangle\}.
\]

By $\mathcal{P}(A)$ we denote the power set of a set $A$, that is, the set of all subsets of $A$:
\[
\mathcal{P}(A) = \{ B : B \subseteq A \}.
\]
For instance,
\[
\mathcal{P}(\{2, 3\}) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}.
\]

Binary Relations

Any condition on a pair of elements of a set $A$ defines a binary relation, or simply relation, on $A$. For instance, the condition $x < y$ defines a relation on the set $\mathbb{N}$ of nonnegative integers (or on any other set of numbers). If $R$ is a relation, the formula $xRy$ expresses that $R$ holds for the pair $x, y$.

A relation $R$ can be characterized by the set of all ordered pairs $\langle x, y \rangle$ such that $xRy$. It is customary to talk about a relation as it were the same thing as the corresponding set of ordered pairs. For instance, we can say that the relation $<$ on the set $\{1, 2, 3, 4\}$ is the set
\[
\{ \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle, \langle 2, 4 \rangle, \langle 3, 4 \rangle \}.
\]
A relation $R$ on a set $A$ is said to be reflexive if, for all elements $x$ of $A$, $xRx$. We say that $R$ is irreflexive if there is no element $x$ of $A$ such that $xRx$. For instance, the relations $=$ and $\leq$ on the set $\mathbb{N}$ (or on any set of numbers) are reflexive, and the relations $\neq$ and $<$ are irreflexive.

A relation $R$ on a set $A$ is said to be symmetric if, for all $x, y \in A$, $xRy$ implies $yRx$. For instance, the relations $=$ and $\neq$ on $\mathbb{N}$ are symmetric, and the relations $<$ and $\leq$ are not.

A relation $R$ on a set $A$ is said to be transitive if, for all $x, y, z \in A$, $xRy$ and $yRz$ imply $xRz$. For instance, the relations $=$, $<$ and $\leq$ on the set $\mathbb{N}$ are transitive, and the relation $\neq$ is not.

**Equivalence Relations and Partitions**

An equivalence relation is a relation that is reflexive, symmetric, and transitive.

A partition of a set $A$ is a collection $P$ of non-empty subsets of $A$ such that every element of $A$ belongs to exactly one of these subsets. For instance, here are some partitions of $\mathbb{N}$:

- $P_1 = \{\{0, 2, 4, \ldots\}, \{1, 3, 5, \ldots\}\}$,
- $P_2 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots\}$,
- $P_3 = \{\{0\}, \{1\}, \{2\}, \{3\}, \ldots\}$.

If $P$ is a partition of a set $A$ then the relation “$x$ and $y$ belong to the same element of $P$” is an equivalence relation.

**Order Relations**

A relation $R$ on a set $A$ is said to be antisymmetric if, for all $x, y \in A$, $xRy$ and $yRx$ imply $x = y$. For instance, the relation $\leq$ on $\mathbb{R}$ is antisymmetric.

A partial order is a relation that is reflexive, anti-symmetric, and transitive. For instance, the relation $\leq$ on $\mathbb{R}$, the relation $|$ on $\mathbb{N}$, and the relation $\subseteq$ on $\mathcal{P}(A)$ for any set $A$ are partial orders.

A total order on a set $A$ is a partial order such that for all $x, y \in A$, $xRy$ or $yRx$. For instance, $\leq$ is total and $|$ is not.

**General Definition of a Function**

For any sets $A$ and $B$, a function from $A$ to $B$ is a rule $f$ that can be applied to any element $x$ of $A$ and produces an element $f(x)$ of $B$. The set $A$ is called the domain of $f$. The subset of $B$ consisting of the values $f(x)$ of the function for all $x \in A$ is called the range of $f$. If the range of $f$ is the whole set $B$ then we say that $f$ is a function onto $B$.

This definition of a function is general because it does not assume that the domain and the range consist of numbers. In the following examples, the domain of each function is the set $S$ of all bit strings:

$$S = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}.$$
1. Function \( l \) from \( S \) to \( N \): \( l(x) \) is the length of \( x \). For instance, 
   \( l(00110) = 5 \).

2. Function \( z \) from \( S \) to \( N \): \( z(x) \) is the number zeroes in \( x \). For instance, 
   \( z(00110) = 3 \).

3. Function \( n \) from \( S \) to \( N \): \( n(x) \) is the number represented by \( x \) in binary notation. 
   For instance, \( n(00110) = 6 \).

4. Function \( e \) from \( S \) to \( S \): \( e(x) \) is the string 1\( x \). For instance, 
   \( e(00110) = 100110 \).

5. Function \( r \) from \( S \) to \( S \): \( r(x) \) is the string \( x \) reversed. For instance, 
   \( r(00110) = 01100 \).

6. Function \( p \) from \( S \) to \( P(S) \): \( p(x) \) is the set of prefixes of \( x \). For instance, 
   \( p(00110) = \{ \epsilon, 0, 00, 001, 0011, 00110 \} \).

A function \( f \) can be characterized by the set of all ordered pairs of the form \( (x, f(x)) \). 
It is customary to talk about a function as if it were the same thing as the corresponding 
set of ordered pairs. For instance, we can say that the function \( f \) from \( N \) to \( N \) defined 
by the formula \( f(n) = 2n + 1 \) is the set 
   \( \{(0, 1), (1, 3), (2, 5), \ldots \} \).

Instead of defining functions as rules, we can say that a function from a set \( A \) to a set \( B \) 
is a set \( f \subseteq A \times B \) such that for every element \( x \) of \( A \) there exists a unique element \( y \)
of \( B \) for which \( (x, y) \in f \).

A function \( f \) from \( A \) to \( B \) is called one-to-one if, for any pair of different elements 
\( x, y \) of \( A \), \( f(x) \) is different from \( f(y) \). If a function \( f \) is both onto and one-to-one then 
we say that \( f \) is a bijection. A permutation of a set \( A \) is a bijection from \( A \) to \( A \).

If \( f \) is a function from \( A \) to \( B \), and \( g \) is a function from \( B \) to \( C \), then the composition 
of these functions is the function \( h \) from \( A \) to \( C \) defined by the formula \( h(x) = g(f(x)) \). 
This function is denoted by \( g \circ f \).

If \( f \) is a bijection from \( A \) to \( B \) then the inverse of \( f \) is the function \( g \) from \( B \) to \( A \) 
such that, for every \( x \in A \), \( g(f(x)) = x \). This function is denoted by \( f^{-1} \).